

BIANCHI IDENTITY FOR THE RIEMANN TENSOR

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Another relation of the Riemann tensor involves the covariant derivative of the tensor, and is known as the Bianchi identity (actually the second Bianchi identity; the first identity is the symmetry relation $R_{nj\ell m} + R_{n\ell m j} + R_{nmj\ell} = 0$ that we saw earlier). The identity is easiest to derive at the origin of a locally inertial frame (LIF), where the first derivatives of the metric tensor, and thus the Christoffel symbols, are all zero. At this point, we have

$$R_{nj\ell m} = \frac{1}{2} (\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}) \quad (1)$$

If the Christoffel symbols are all zero, then the covariant derivative becomes the ordinary derivative

$$\nabla_j A^k \equiv \partial_j A^k + A^i \Gamma_{ij}^k = \partial_j A^k \quad (2)$$

Therefore, we get, at the origin of a LIF:

$$\nabla_k R_{nj\ell m} = \partial_k R_{nj\ell m} \quad (3)$$

$$= \frac{1}{2} (\partial_k \partial_\ell \partial_j g_{mn} + \partial_k \partial_m \partial_n g_{j\ell} - \partial_k \partial_\ell \partial_n g_{jm} - \partial_k \partial_m \partial_j g_{\ell n}) \quad (4)$$

By cyclically permuting the index of the derivative with the last two indices of the tensor, we get

$$\nabla_\ell R_{njmk} = \partial_\ell R_{njmk} \quad (5)$$

$$= \frac{1}{2} (\partial_\ell \partial_m \partial_j g_{kn} + \partial_\ell \partial_k \partial_n g_{jm} - \partial_\ell \partial_m \partial_n g_{jk} - \partial_\ell \partial_k \partial_j g_{mn}) \quad (6)$$

$$\nabla_m R_{njkl} = \partial_m R_{njkl} \quad (7)$$

$$= \frac{1}{2} (\partial_m \partial_k \partial_j g_{\ell n} + \partial_m \partial_\ell \partial_n g_{jk} - \partial_m \partial_k \partial_n g_{j\ell} - \partial_m \partial_\ell \partial_j g_{kn}) \quad (8)$$

By adding up 4, 6 and 8 and using the commutativity of partial derivatives, we see that the terms cancel in pairs, so we get

$$\boxed{\nabla_k R_{nj\ell m} + \nabla_\ell R_{njmk} + \nabla_m R_{njkl} = 0} \quad (9)$$

As usual we can use the argument that since we can set up a LIF with its origin at any non-singular point in spacetime, this equation is true everywhere and since the covariant derivative is a tensor, this is a tensor equation and is thus valid in all coordinate systems. This is the Bianchi identity.

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