BOYER-LINDQUIST COORDINATES AND CURVATURE OF SPACE

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In Appendix I.5.2 of Zee's book, he discusses a modification of the spherical coordinate metric in flat space with the definitions

$$\begin{aligned} x &= f(r)\sin\theta\cos\phi \\ y &= f(r)\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned} \tag{1}$$

where $f(r)$ is some function to be specified.

We can write the usual Euclidean metric in terms of these new coordinates by finding the differentials:

$$\begin{aligned} dx &= f'(r)\sin\theta\cos\phi\, dr + f(r)\cos\theta\cos\phi\, d\theta - f(r)\sin\theta\sin\phi\, d\phi \\ dy &= f'(r)\sin\theta\sin\phi\, dr + f(r)\cos\theta\sin\phi\, d\theta + f(r)\sin\theta\cos\phi\, d\phi \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta \end{aligned} \tag{2}$$

We can now write the distance element by calculating (using Maple to simplify things):

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

$$= \left(f'^2\sin^2\theta + \cos^2\theta\right)dr^2 + \left(f^2\cos^2\theta + r^2\sin^2\theta\right)d\theta^2 + f^2\sin^2\theta\, d\phi^2 + 2\left(ff' - r\right)\sin\theta\cos\theta\, dr\, d\theta \tag{4}$$

With an arbitrary f, this metric is rather ugly in that it's not even diagonal due to the presence of the $dr\, d\theta$ term. We can get rid of this term by choosing

$$ff' = r \tag{5}$$

which gives us

$$f\, df = r\, dr \tag{6}$$

which integrates to give

$$f^2 = r^2 + a^2 \tag{7}$$

where a is a constant of integration.

With this substitution, the metric becomes (again, after simplifying with Maple):

$$ds^2 = \frac{r^2 + a^2\cos^2\theta}{a^2 + r^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta\, d\phi^2 \tag{8}$$

This description of flat space is known as the Boyer-Linquist coordinate system. Choosing $a = 0$ restores the standard spherical coordinate system, since then $f = r$.

In exercise I.5.4 of Zee's book, we are given the metric (note that the equation in Zee's book is missing a factor of $\sin^2\theta$):

$$ds^2 = \frac{\rho^2}{\rho^2 + a^2\sin^2\theta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 \tag{9}$$

This is the same metric as 8, with

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{10}$$

We have

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi \tag{11}$$

$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \tag{12}$$

$$z = r\cos\theta \tag{13}$$

For $a \neq 0$, if we set $r = 0$ then $z = 0$ so we're restricted to the xy plane. However, we still have a range of values possible for x and y, so $r = 0$ describes a disc rather than a single point. For a fixed value of θ, the point (x, y) is a point on this disc with radius $\sin\theta$. As we vary ϕ, this point describes a circle around the origin. Thus as $\sin\theta$ increases from 0 to 1, we have successively larger circles up to a circle with radius a.

To investigate the curvature of this metric, we can follow the procedure in Zee's Appendix I.5.1, where we evaluate the curvature given by R, defined as

$$R = \lim_{\text{radius}\to 0}\frac{6}{\text{radius}^2}\left(1 - \frac{\text{circumference}}{2\pi\text{radius}}\right) \tag{14}$$

where 'circumference' and 'radius' refer to a circle drawn around the point at which we wish to find the curvature.

Consider the origin in our current example. The distance from the origin to a circle of radius $\sin\epsilon$ is the distance measured along a curve of constant θ (since θ describes the radius as explained above). So

$$\text{radius } = \int_0^{\epsilon} \rho(r=0,\theta)\,d\theta = a\int_0^{\epsilon}\cos\theta\,d\theta \tag{15}$$

$$= a\sin\epsilon \tag{16}$$

The circumference of this circle is the length of the curve with radius $a\sin\epsilon$ as ϕ goes from 0 to 2π. That is

$$\text{circumference} = \int_0^{2\pi} a\sin\epsilon\,d\phi = 2\pi a\sin\epsilon \tag{17}$$

The curvature from 14 is then

$$R = \lim_{\epsilon\to 0}\frac{6}{a^2\sin^2\epsilon}\left(1-\frac{2\pi a\sin\epsilon}{2\pi a\sin\epsilon}\right) = 0 \tag{18}$$

Thus the space is flat at the origin.

To study lines of fixed θ and ϕ (I'm assuming he means having both these fixed at the same time), we can play with the defining equations 12. We have

$$r = \frac{z}{\cos\theta} \tag{19}$$

so

$$x^2 = \left(\frac{z^2}{\cos^2\theta} + a^2\right)\sin^2\theta\cos^2\phi \tag{20}$$

$$y^2 = \left(\frac{z^2}{\cos^2\theta} + a^2\right)\sin^2\theta\sin^2\phi \tag{21}$$

Therefore

$$\frac{x^2}{\cos^2\phi} - \frac{y^2}{\sin^2\phi} = 0 \tag{22}$$

$$y = \pm x\tan\phi \tag{23}$$

Thus for a fixed value of ϕ, the plot of y versus x is a straight line.

We also have

$$\frac{z^2}{\cos^2\theta} = \frac{x^2+y^2}{\sin^2\theta} - a^2 \tag{24}$$

$$= \frac{x^2}{\sin^2\theta}\left(1+\tan^2\phi\right) - a^2 \tag{25}$$

$$= \frac{x^2}{\sin^2\theta\cos^2\phi} - a^2 \tag{26}$$

Rearranging, we have

$$\frac{x^2}{\sin^2\theta\cos^2\phi} - \frac{z^2}{\cos^2\theta} = a^2 \tag{27}$$

For fixed θ, ϕ and a this is the equation of a hyperbola, with asymptotes given by

$$x = \pm\frac{\tan\theta}{\cos\phi}z \tag{28}$$

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