CHRISTOFFEL SYMBOLS - SYMMETRY

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The Christoffel symbols are defined in terms of the basis vectors in a given coordinate system as:

$$\boxed{\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{e}_k} \tag{1}$$

Remember that the basis vectors $\mathbf{e}_i$ are defined so that

$$\begin{aligned}
ds^2 &= d\mathbf{s} \cdot d\mathbf{s} && (2)\\
&= \left(dx^i \mathbf{e}_i\right) \cdot \left(dx^j \mathbf{e}_j\right) && (3)\\
&= \mathbf{e}_i \cdot \mathbf{e}_j dx^i dx^j && (4)\\
&\equiv g_{ij} dx^i dx^j && (5)
\end{aligned}$$

In a locally flat frame using rectangular spatial coordinates, the basis vectors $\mathbf{e}_i$ are all constants, so from 1, all the Christoffel symbols must be zero: $\Gamma_{ij}^k = 0$.

Now let's look at the second covariant derivative of a scalar field Φ:

$$\begin{aligned}
\nabla_i \nabla_j \Phi &= \nabla_i \left(\partial_j \Phi\right) && (6)\\
&= \partial_i \partial_j \Phi - \Gamma_{ij}^k \partial_k \Phi && (7)
\end{aligned}$$

where in 6 we used rule 1 for the covariant derivate: the covariant derivative of a scalar is the same as the ordinary derivative.

In the locally flat frame, this equation reduces to

$$\nabla_i \nabla_j \Phi = \partial_i \partial_j \Phi \tag{8}$$

Since the covariant derivative is a tensor, this is a tensor equation, and since ordinary partial derivatives commute, this equation is the same if we swap the indices i and j. Tensor equations must have the same form in all coordinate systems, so this implies that 7 must also be invariant if we swap i and j. This means that the Christoffel symbols are symmetric under exchange of their two lower indices:

$$\boxed{\Gamma_{ij}^{k}=\Gamma_{ji}^{k}} \tag{9}$$

At first glance, this seems wrong, since from the definition 1 this symmetry implies that

$$\frac{\partial\mathbf{e}_{i}}{\partial x^{j}}=\frac{\partial\mathbf{e}_{j}}{\partial x^{i}} \tag{10}$$

In 2-D polar coordinates, if we take the usual unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ then both these vectors are constants as we change r and both of them change when we change θ, so it's certainly not true that $\partial\hat{\mathbf{r}}/\partial\theta=\partial\hat{\boldsymbol{\theta}}/\partial r$, for example. However, remember that the basis vectors we're using are *not* the usual unit vectors; rather they are defined so that condition 4 is true. In polar coordinates, we have

$$ds^{2}=dr^{2}+r^{2}d\theta^{2} \tag{11}$$

so

$$\mathbf{e}_{r}=\hat{\mathbf{r}}=\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}} \tag{12}$$

$$\mathbf{e}_{\theta}=r\hat{\boldsymbol{\theta}}=-r\sin\theta\hat{\mathbf{x}}+r\cos\theta\hat{\mathbf{y}} \tag{13}$$

For the derivatives, we have

$$\frac{\partial\mathbf{e}_{r}}{\partial\theta}=-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}=\hat{\boldsymbol{\theta}} \tag{14}$$

$$\frac{\partial\mathbf{e}_{\theta}}{\partial r}=-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}=\hat{\boldsymbol{\theta}} \tag{15}$$

Thus the condition 10 is actually satisfied here.

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