CHRISTOFFEL SYMBOLS AND THE COVARIANT DERIVATIVE

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It's time to return to the study of tensors so we can pave the way for the Einstein equation which is the basis of general relativity. Up to now, we've assumed that the Schwarzschild metric described spacetime outside a spherical mass, without any arguments to back up this assumption. Developing these arguments requires some more tools from the tensor toolbox.

One problem with tensors is that their straightforward derivatives are not, in general, tensors themselves. For example, the first derivative of a scalar function f is a covariant tensor, as it transforms according to

$$\frac{\partial f}{\partial x'^a}=\frac{\partial x^i}{\partial x'^a}\frac{\partial f}{\partial x^i} \tag{1}$$

If we take the derivative of this equation with respect to another of the primed coordinates x'^c, we get

$$\begin{aligned}\frac{\partial^2 f}{\partial x'^a\partial x'^c} &= \frac{\partial^2 x^i}{\partial x'^a\partial x'^c}\frac{\partial f}{\partial x^i}+\frac{\partial x^i}{\partial x'^a}\frac{\partial^2 f}{\partial x^i\partial x'^c} &(2)\\ &= \frac{\partial^2 x^i}{\partial x'^a\partial x'^c}\frac{\partial f}{\partial x^i}+\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^c}\frac{\partial^2 f}{\partial x^i\partial x^j} &(3)\end{aligned}$$

where we used the chain rule on the second term. In order for $\frac{\partial^2 f}{\partial x^i\partial x^j}$ to transform like a tensor, the first term in the last line would have to be zero, but it's clearly not, in general.

To remedy this problem, first cast your mind back to the definition of basis vectors in some coordinate system. The basis vectors $\mathbf{e}_i$ are linearly independent vectors that are tangent to the lines of constant coordinate values and that span the space. They must satisfy

$$\begin{aligned}ds^2 &= d\mathbf{s}\cdot d\mathbf{s} &(4)\\ &= \left(dx^i\mathbf{e}_i\right)\cdot\left(dx^j\mathbf{e}_j\right) &(5)\\ &= \mathbf{e}_i\cdot\mathbf{e}_j dx^i dx^j &(6)\\ &\equiv g_{ij}dx^i dx^j &(7)\end{aligned}$$

where g_{ij} is the metric tensor. Keep in mind that, for a general coordinate system, these basis vectors need not be either orthogonal or unit vectors, and that they can change as we move around. As such, we can consider the derivative of basis vector $\mathbf{e}_i$ with respect to coordinate x^j with all other coordinates held constant. Since the derivative of a vector is another vector, and the basis vectors span the space, we can express this derivative as a linear combination of the basis vectors at the point at which the derivative is taken. That is

$$\boxed{\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{e}_k} \tag{8}$$

The quantities Γ_{ij}^k are called *Christoffel symbols* or *connection coefficients*, named after Elwin Bruno Christoffel, a 19th century German mathematician and physicist. (Students of GR often refer to them as the 'Christawful' symbols, since formulas involving them can be tricky to use and remember due to the number of indices involved.) It's important to note that although Γ_{ij}^k is written with indices that make it look like a tensor, it is **not** a tensor on its own. The transformation equation for Γ_{ij}^k can be derived from its explicit form (which we haven't got to yet), but for reference here it is:

$$\left(\Gamma'\right)^l_{\ mn} = \Gamma_{ij}^k \frac{\partial x'^l}{\partial x^k} \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^n} + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} \tag{9}$$

The first term is the normal transformation for a rank-3 tensor, but the second term spoils the transformation. However, notice that if we use the special coordinate transformation where $x'^m = x^i$ and $x'^n = x^j$, then the second derivative in the second term vanishes, since x^k, x^i and x^j are independent variables in the same coordinate system, so none of them depends on the others. Furthermore, in this special case $\frac{\partial x^i}{\partial' x^m} = \frac{\partial x^j}{\partial x'^n} = 1$, so the transformation becomes

$$\left(\Gamma'\right)^l_{\ ij} = \Gamma_{ij}^k \frac{\partial x'^l}{\partial x^k} \tag{10}$$

which is a valid tensor transformation. That is, the Christoffel symbol's upper index does transform as a tensor, which makes sense, since 8 defines a four-vector with components Γ_{ij}^k, holding i and j fixed.

We'll get to methods for calculating Christoffel symbols in a later post. For now, we'll see how we can define a derivative of a tensor that is itself always another tensor. Suppose we have a vector field $\mathbf{A}\left(x^i\right)$. In a given coordinate system, we can write this in terms of the basis vectors:

$$\mathbf{A} = A^i \mathbf{e}_i \tag{11}$$

If we calculate its differential we get

$$\begin{aligned}
d\mathbf{A} &= d\left(A^i \mathbf{e}_i\right) && (12)\\
&= \left(dA^i\right)\mathbf{e}_i + A^i\left(d\mathbf{e}_i\right) && (13)\\
&= \left(\frac{\partial A^i}{\partial x^j}dx^j\right)\mathbf{e}_i + A^i\left(\frac{\partial \mathbf{e}_i}{\partial x^j}dx^j\right) && (14)\\
&= \left(\frac{\partial A^i}{\partial x^j}dx^j\right)\mathbf{e}_i + A^i\Gamma_{ij}^k\mathbf{e}_k dx^j && (15)\\
&= \left[\frac{\partial A^k}{\partial x^j} + A^i\Gamma_{ij}^k\right]\mathbf{e}_k dx^j && (16)\\
&= \nabla_j A^k \mathbf{e}_k dx^j && (17)
\end{aligned}$$

where in line 16 we relabelled the dummy summation index i to k in the first term, and in line 17 we defined the *covariant derivative* (sometimes called the *absolute gradient*):

$$\boxed{\nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i\Gamma_{ij}^k} \tag{18}$$

The combination $\nabla_j A^k dx^j$ has only one free index k, and it appears in the form

$$\left(\nabla_j A^k dx^j\right)\mathbf{e}_k \tag{19}$$

so that $\nabla_j A^k dx^j$ is component k of a four-vector. Since the differential dx^j is a tensor, the absolute gradient $\nabla_j A^k$ must also be a tensor of rank 2. We've thus found a derivative of a tensor (well, just a four-vector so far) that is itself a tensor.

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