CHRISTOFFEL SYMBOLS DEFINED FOR A SPHERE

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Our original derivation of the Christoffel symbols was in terms of the derivatives of the basis vectors for a manifold. We had

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^k_{ij}\mathbf{e}_k \tag{1}$$

where $\boldsymbol{e}_i$ are the basis vectors. The rationale for this is that, in a general coordinate system or curved manifold, the basis vectors change as we move around, so they have non-zero derivatives. We also stated that, because the derivative of a basis vector is another vector then it must be expressible as a linear combination of the basis vectors for that manifold. We illustrated this by considering the case of polar coordinates in the plane.

This derivation is a bit dishonest, since using polar coordinates introduces a curved coordinate system, but in a flat space. In this case, the two basis vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ actually do span the entire plane, no matter which point we use to define the basis vectors.

This simplified case appears to break down if we consider a space that is intrinsically curved, rather that the flat xy plane. To see the problem, consider the case of the surface of a unit sphere. In this case, the tangent space at each point on the sphere is the tangent plane, and the basis vectors for this plane lie within that plane. When we move to a different point on the sphere, the tangent plane is no longer parallel to the original tangent plane, so the basis vectors in the new plane will not, in general, be expressible as linear combination of the original basis vectors. In other words, the basis vectors in the new plane will usually have a component parallel to the normal to the original plane. Thus the derivative of a basis vector must be expressed as

$$\frac{\partial \boldsymbol{e}_i}{\partial x^j} = \Gamma^k{}_{ij}\boldsymbol{e}_k + K_{ij}\boldsymbol{n} \tag{2}$$

where $\boldsymbol{n}$ is the normal vector to the original plane and the K_{ij} are coefficients that depend on the location on the sphere.

We can work out the case of the unit sphere in detail to see how this works. A general point $\boldsymbol{X}$ on the unit sphere is

$$\boldsymbol{X}=\left[\begin{array}{c} s_\theta c_\phi \\ s_\theta s_\phi \\ c_\theta \end{array}\right] \tag{3}$$

where we're using the usual polar angles θ (measured from the north pole) and ϕ (measured counterclockwise from the x axis). [As there are a lot of sines and cosines in what follows, I'm using the shorthand $s_\theta \equiv \sin\theta$ etc to save writing.]

At a point (θ,ϕ) on the sphere, the basis vectors are give by

$$\begin{aligned} \boldsymbol{e}_\theta &= \frac{\partial \boldsymbol{X}}{\partial\theta}=\left[\begin{array}{c} c_\theta c_\phi \\ c_\theta s_\phi \\ -s_\theta \end{array}\right] \\ \boldsymbol{e}_\phi &= \frac{\partial \boldsymbol{X}}{\partial\phi}=\left[\begin{array}{c} -s_\theta s_\phi \\ s_\theta c_\phi \\ 0 \end{array}\right] \end{aligned} \tag{4}$$

These two vectors (note that although $\boldsymbol{e}_\theta$ is a unit vector, $\boldsymbol{e}_\phi$ is not) lie in the tangent plane at the point (θ,ϕ), so their cross product is normal to the tangent plane.

$$\boldsymbol{e}_\theta\times\boldsymbol{e}_\phi = \left|\begin{array}{ccc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ c_\theta c_\phi & c_\theta s_\phi & -s_\theta \\ -s_\theta s_\phi & s_\theta c_\phi & 0 \end{array}\right| \tag{5}$$

$$= \left[\begin{array}{c} s_\theta^2 c_\phi \\ s_\theta^2 s_\phi \\ s_\theta c_\theta \end{array}\right] \tag{6}$$

We now define $\boldsymbol{n}$ to be the unit normal, so we have

$$\boldsymbol{n}=\frac{\boldsymbol{e}_\theta\times\boldsymbol{e}_\phi}{\left|\boldsymbol{e}_\theta\times\boldsymbol{e}_\phi\right|}=\left[\begin{array}{c} s_\theta c_\phi \\ s_\theta s_\phi \\ c_\theta \end{array}\right] \tag{7}$$

We can now take the derivatives of 4 and compare them with 2 to find ${\Gamma^k}_{ij}$ and K_{ij}. We have, using the notation $\boldsymbol{e}_{i,j} \equiv \frac{\partial \boldsymbol{e}_i}{\partial x^j}$:

$$\begin{aligned}\boldsymbol{e}_{\theta,\theta} &= \begin{bmatrix} -s_\theta c_\phi \\ -s_\theta s_\phi \\ -c_\theta \end{bmatrix} \\ \boldsymbol{e}_{\phi,\theta} = \boldsymbol{e}_{\theta,\phi} &= \begin{bmatrix} -c_\theta s_\phi \\ c_\theta c_\phi \\ 0 \end{bmatrix} \\ \boldsymbol{e}_{\phi,\phi} &= \begin{bmatrix} -s_\phi c_\phi \\ -s_\theta s_\phi \\ 0 \end{bmatrix}\end{aligned} \tag{8}$$

[Note that $\boldsymbol{e}_{\phi,\theta} = \boldsymbol{e}_{\theta,\phi}$ follows from 4 since the order of the derivatives commutes:

$$\boldsymbol{e}_{\phi,\theta} = \frac{\partial^2 \boldsymbol{X}}{\partial\theta\partial\phi} = \frac{\partial^2 \boldsymbol{X}}{\partial\phi\partial\theta} = \boldsymbol{e}_{\theta,\phi} \tag{9}$$

Comparing with 2 and 7 we have

$$\boldsymbol{e}_{\theta,\theta} = \begin{bmatrix} -s_\theta c_\phi \\ -s_\theta s_\phi \\ -c_\theta \end{bmatrix} = -\boldsymbol{n} \tag{10}$$

Thus from $\boldsymbol{e}_{\theta,\theta}$

$$\begin{aligned}\Gamma^\theta{}_{\theta\theta} = \Gamma^\phi{}_{\theta\theta} &= 0 \\ K_{\theta\theta} &= -1\end{aligned} \tag{11}$$

From $\boldsymbol{e}_{\theta,\phi}$ or $\boldsymbol{e}_{\phi,\theta}$

$$\boldsymbol{e}_{\phi,\theta} = \boldsymbol{e}_{\theta,\phi} = \begin{bmatrix} -c_\theta s_\phi \\ c_\theta c_\phi \\ 0 \end{bmatrix} = \frac{c_\theta}{s_\theta}\boldsymbol{e}_\phi \tag{12}$$

so we have

$$\begin{aligned}\Gamma^\theta{}_{\phi\theta} &= \Gamma^\theta{}_{\theta\phi} = 0 \\ \Gamma^\phi{}_{\phi\theta} &= \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} = \cot\theta \\ K_{\theta\phi} &= K_{\phi\theta} = 0\end{aligned} \tag{13}$$

For $\boldsymbol{e}_{\phi,\phi}$ it's not quite as simple, but substituting from 8 into 2 and 7 we get a system of three equations:

$$\boldsymbol{e}_{\phi,\phi} = a\boldsymbol{e}_\theta + b\boldsymbol{e}_\phi + c\boldsymbol{n} \tag{14}$$

or

$$\begin{aligned}
-s_\theta c_\phi &= ac_\theta c_\phi - bs_\theta s_\phi + cs_\theta c_\phi \\
-s_\theta s_\phi &= ac_\theta s_\phi + bs_\theta c_\phi + cs_\theta s_\phi \\
0 &= -as_\theta + cc_\theta
\end{aligned} \tag{15}$$

and we must find a, b and c. This can be done by hand, although I used Maple and we find

$$\begin{aligned}
a &= -\frac{c_\theta s_\theta}{c_\theta^2 + s_\theta^2} = -\sin\theta\cos\theta \\
b &= 0 \\
c &= -\frac{s_\theta^2}{c_\theta^2 + s_\theta^2} = -\sin^2\theta
\end{aligned} \tag{16}$$

Thus we have

$$\begin{aligned}
a &= \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \\
b &= \Gamma^{\phi}{}_{\phi\phi} = 0 \\
c &= K_{\phi\phi} = -\sin^2\theta
\end{aligned} \tag{17}$$

Putting it all together, we have

$$\begin{aligned}
\boldsymbol{e}_{\theta,\theta} &= -\boldsymbol{n} \\
\boldsymbol{e}_{\theta,\phi} = \boldsymbol{e}_{\phi,\theta} &= \cot\theta\boldsymbol{e}_\phi \\
\boldsymbol{e}_{\phi,\phi} &= -\sin\theta\cos\theta\boldsymbol{e}_\theta - \sin^2\theta\boldsymbol{n}
\end{aligned} \tag{18}$$

Thus we find that, when viewed in the 3-dim space in which the unit sphere is embedded, 1 is missing the components that are normal to the tangent plane. However, if we restrict ourselves to the surface of the sphere, we can define the covariant derivative as the derivative that considers only changes to the basis vectors that lie in the tangent surface as it moves around the sphere. That is, we ignore any components that are in the direction of $\boldsymbol{n}$, which then does restore the definition 1. That is, we get

$$\begin{aligned}
\boldsymbol{e}_{\theta,\theta} &= 0 \\
\boldsymbol{e}_{\theta,\phi} = \boldsymbol{e}_{\phi,\theta} &= \cot\theta\boldsymbol{e}_\phi \\
\boldsymbol{e}_{\phi,\phi} &= -\sin\theta\cos\theta\boldsymbol{e}_\theta
\end{aligned} \tag{19}$$

Geometrically, we can test a few cases to see if this makes sense. The geodesics on a sphere are the great circles, with diameters equal to the diameter of the sphere. A line of longitude is a geodesic, and along such a line, the vector $\boldsymbol{e}_\theta$ always points due south and maintains its unit length. Thus the condition $\boldsymbol{e}_{\theta,\theta} = 0$ makes sense the point of view of someone confined to the sphere's surface. The vector is parallel transported along the geodesic and doesn't change from this viewpoint, even though, when viewed in the 3-dim embedding space, the vector *does* change direction.

The only line of latitude that is a geodesic is the equator where $\theta = \frac{\pi}{2}$. Along this line, $\boldsymbol{e}_{\phi,\phi} = 0$ since again the vector $\boldsymbol{e}_\phi$ is parallel transported around the equator. In the embedding space, there is also a component of $\boldsymbol{e}_{\phi,\phi}$ along $\boldsymbol{n}$ to account for the change of direction as we move along the equator.

Although this derivation was done for the specific case of a 2-dim curved surface embedded in 3-dim space, we can generalize it to any number of dimensions (specifically the 4 dimensions of spacetime) by defining the derivative of a basis vector to be given by 1, that is, we ignore the contributions to the derivative from any components that don't lie in the tangent space.