CHRISTOFFEL SYMBOLS FOR WORMHOLE METRIC

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We can find the Christoffel symbols for the wormhole metric by the usual method of comparing the two forms of the geodesic equation. The wormhole metric is

$$ds^2 = -dt^2 + dr^2 + \left(b^2 + r^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

where b is a constant.

To find the Christoffel symbols, we follow the same procedure as for the Schwarzchild metric. The geodesic equation is (where a dot above a symbol means the derivative with respect to τ):

$$g_{\alpha\nu}\ddot{x}^\nu + \left(\partial_\mu g_{\alpha\nu} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\right)\dot{x}^\nu \dot{x}^\mu = 0 \tag{2}$$

The following equation is formally equivalent to this:

$$\ddot{x}^\beta + \Gamma^\beta{}_{\mu\nu}\dot{x}^\nu \dot{x}^\mu = 0 \tag{3}$$

The method for calculating the Christoffel symbols is to work out the terms in 2, divide through by $g_{\alpha\mu}$ and then compare the result term by term with 3. By doing this we are able to read off the $\Gamma^\beta{}_{\mu\nu}$ as the coefficients of $\dot{x}^\nu \dot{x}^\mu$ in 2.

Equation 2 is actually four equations, one for each value of α. The indices μ and ν are summed. From 1 we have

$$\begin{aligned} g_{tt} &= -1 \\ g_{rr} &= 1 \\ g_{\theta\theta} &= b^2 + r^2 \\ g_{\phi\phi} &= \left(b^2 + r^2\right)\sin^2\theta \end{aligned} \tag{4}$$

with all off-diagonal elements $g_{\mu\nu}$ equal to zero. From 2 we have, for $\alpha = t$ (no sum over t):

$$g_{tt}\ddot{t} + \left(\partial_t g_{tt} - \frac{1}{2}\partial_t g_{\mu\nu}\right)\dot{x}^\nu \dot{x}^\mu = 0 \tag{5}$$

The second term is zero because the metric is independent of t. Thus we have

$$\frac{d^2t}{d\tau^2}=0 \tag{6}$$

For $\alpha=r$, we have

$$g_{rr}\frac{d^2r}{d\tau^2}+\left(\partial_r g_{rr}-\frac{1}{2}\partial_r g_{\mu\nu}\right)\dot{x}^\nu\dot{x}^\mu=0 \tag{7}$$

From 4, we see that $\partial_r g_{rr}=0$. The second term in parentheses can be summed so we have, using $g_{rr}=1$:

$$\frac{d^2r}{d\tau^2}-\frac{1}{2}2r\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=0 \tag{8}$$

$$\frac{d^2r}{d\tau^2}-r\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=0 \tag{9}$$

For $\alpha=\theta$, we have

$$\left(b^2+r^2\right)\ddot{\theta}+\partial_\mu g_{\theta\theta}\dot{\theta}\dot{x}^\mu-\frac{1}{2}\partial_\theta g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu=0 \tag{10}$$

Breaking this down, we have

$$\partial_\mu g_{\theta\theta}\dot{\theta}\dot{x}^\mu=2r\frac{d\theta}{d\tau}\frac{dr}{d\tau} \tag{11}$$

$$-\frac{1}{2}\partial_\theta g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu=-\frac{1}{2}\left(b^2+r^2\right)(2\sin\theta\cos\theta)\left(\frac{d\phi}{d\tau}\right)^2 \tag{12}$$

$$=-\left(b^2+r^2\right)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{13}$$

Putting it together, we get

$$\left(b^2+r^2\right)\frac{d^2\theta}{d\tau^2}=-2r\frac{d\theta}{d\tau}\frac{dr}{d\tau}+\left(b^2+r^2\right)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{14}$$

or

$$\frac{d^2\theta}{d\tau^2}+\frac{2r}{b^2+r^2}\frac{d\theta}{d\tau}\frac{dr}{d\tau}-\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2=0 \tag{15}$$

Finally, for $\alpha=\phi$ we have

$$\left(b^2+r^2\right)\sin^2\theta\ddot{\phi}+\partial_\mu g_{\phi\phi}\dot{\phi}\dot{x}^\mu-\frac{1}{2}\partial_\phi g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu=0 \tag{16}$$

Because the metric is independent of ϕ, the last term is zero, and we have

$$\left(b^2+r^2\right)\sin^2\theta\frac{d^2\phi}{d\tau^2}+2r\sin^2\theta\frac{d\phi}{d\tau}\frac{dr}{d\tau}+\left(b^2+r^2\right)2\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}=0 \tag{17}$$

or

$$\frac{d^2\phi}{d\tau^2}+\frac{2r}{b^2+r^2}\frac{d\phi}{d\tau}\frac{dr}{d\tau}+2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}=0 \tag{18}$$

We can now compare each of 6, 9, 15 and 18 with 3 to read off the Christoffel symbols. The non-zero symbols are

$$\begin{aligned}
\Gamma^r{}_{\theta\theta} &= -r\\
\Gamma^r{}_{\phi\phi} &= -r\sin^2\theta\\
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{r}{b^2+r^2}\\
\Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta\\
\Gamma^\phi{}_{\phi r} = \Gamma^\phi{}_{r\phi} &= \frac{r}{b^2+r^2}\\
\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} &= \frac{\cos\theta}{\sin\theta} = \cot\theta
\end{aligned} \tag{19}$$

The results for $\Gamma^\theta{}_{r\theta}=\Gamma^\theta{}_{\theta r}$ and $\Gamma^\phi{}_{\theta\phi}=\Gamma^\phi{}_{\phi\theta}$ follow from the fact that in the sum over μ and ν in 3 we have two of each off-diagonal term. For example, we have $\Gamma^\theta{}_{r\theta}\dot{r}\dot{\theta}+\Gamma^\theta{}_{\theta r}\dot{\theta}\dot{r}=2\Gamma^\theta{}_{r\theta}\dot{r}\dot{\theta}$ due to the symmetry of $\Gamma^\theta{}_{r\theta}$. When comparing this with 15, we have

$$2\Gamma^\theta{}_{r\theta}=2\Gamma^\theta{}_{\theta r}=\frac{2r}{b^2+r^2} \tag{20}$$