CHRISTOFFEL SYMBOLS IN NONCOORDINATE BASES

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Post date: 26 Dec 2022.

A noncoordinate basis is a basis that has no corresponding coordinate system. In a coordinate basis, the one-form of a scalar field ϕ gives the gradient of that scalar, so we have

$$\tilde{d}\phi\left(\vec{e}_{\mu}\right)=\phi_{,\mu}=\frac{\partial\phi}{\partial x^{\mu}} \tag{1}$$

where $\vec{e}_{\mu}$ is the μth basis vector in a coordinate basis, with corresponding coordinates x^{μ}.

If we denote a noncoordinate basis by putting a hat over the index, then $\vec{e}_{\hat{\mu}}$ is a basis vector in such a system. For polar coordinates, the noncoordinate basis is defined by

$$\begin{aligned}\vec{e}_{\hat{r}}&=\cos\theta\vec{e}_{x}+\sin\theta\vec{e}_{y}\\ \vec{e}_{\hat{\theta}}&=\frac{1}{r}\hat{e}_{\theta}=-\sin\theta\vec{e}_{x}+\cos\theta\vec{e}_{y}\end{aligned} \tag{2}$$

In section 5.5, Schutz defines the quantity $\phi_{,\hat{\mu}}$ by

$$\phi_{,\hat{\mu}}=\tilde{d}\phi\left(\vec{e}_{\hat{\mu}}\right) \tag{3}$$

and says that, in polar coordinates

$$\phi_{,\hat{\theta}}=\frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{4}$$

It would seem that what he has done is insert 2 into 3 and apply 1, keeping r constant:

$$\tilde{d}\phi\left(\vec{e}_{\hat{\theta}}\right)=\frac{\partial}{\partial\theta}\left(\frac{1}{r}\phi\right)=\frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{5}$$

I'm not entirely clear on the motivation for such a definition, but comments welcome.

In any case, if we have a transformation matrix $\Lambda^{\beta}{}_{\hat{\alpha}}$ for converting coordinate basis vectors into noncoordinate basis vectors, then the general definition is

$$\nabla_{\hat{\alpha}}\phi \equiv \phi_{,\hat{\alpha}} = \Lambda^{\beta}{}_{\hat{\alpha}}\nabla_{\beta}\phi = \Lambda^{\beta}{}_{\hat{\alpha}}\frac{\partial\phi}{\partial x^{\beta}} \tag{6}$$

Here, the unhatted index β refers to a coordinate system, and that hatted index $\hat{\alpha}$ refers to the noncoordinate basis. For example, to convert from the polar coordinate basis to the noncoordinate basis (in which the basis vectors are orthonormal), we have, by comparison with 2:

$$\Lambda^{\beta}{}_{\hat{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-1} \end{bmatrix} \tag{7}$$

Although a noncoordinate basis doesn't allow a corresponding coordinate system, the quantity $\nabla_{\hat{\beta}}\vec{e}_{\hat{\alpha}}$still gives another vector which, because the $\{\vec{e}_{\hat{\alpha}}\}$ form a basis, can be written as a linear combination of these basis vectors. That is, we can write

$$\nabla_{\hat{\beta}}\vec{e}_{\hat{\alpha}} = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\vec{e}_{\hat{\mu}} \tag{8}$$

where the $\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}$ are the Christoffel symbols in the noncoordinate basis.

If we review the derivation of the symmetry of the Christoffel symbols (that is, that $\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha}$), we see that this proof relied on the fact that a mixed second derivative can be taken in either order, and that the gradient $\nabla_{\mu}\phi$ was just the μ component of the gradient, as in 1. In a noncoordinate basis, the definition 6 shows that this is no longer true in general, so we can't always conclude that Christoffel symbols in such a basis are symmetric.

As we saw in the original derivation of the symmetry, this implies that, in a coordinate basis

$$\nabla_{\alpha}\vec{e}_{\beta} = \nabla_{\beta}\vec{e}_{\alpha} \tag{9}$$

where I've rewritten the equation using Schutz's notation.

In a noncoordinate basis, we can therefore define the deviation from the coordinate basis by considering

$$\nabla_{\mu}\vec{e}_{\nu} - \nabla_{\nu}\vec{e}_{\mu} = c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} \tag{10}$$

[Here I've dropped the hats on the indexes to save some typing, but from here on we're dealing only with noncoordinate bases.] Here the $c^{\alpha}{}_{\mu\nu}$ are the coefficients of the basis vectors. In a coordinate basis, all the $c^{\alpha}{}_{\mu\nu}$ are zero.

We can use this to derive a new condition giving the Christoffel symbols in terms of the metric. The derivation follows a similar method to that used earlier for a coordinate basis, but I've rewritten it to use Schutz's notation.

We first use the fact (see Schutz, section 5.4) that the covariant derivative of the metric is always zero. We then have

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}{}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}{}_{\beta\mu} g_{\alpha\nu} = 0 \tag{11}$$

The covariant derivative for the metric can be written three ways by permuting the indices. We have

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}{}_{\alpha\mu} g_{\nu\beta} + \Gamma^{\nu}{}_{\beta\mu} g_{\alpha\nu} \tag{12}$$
$$g_{\alpha\mu,\beta} = \Gamma^{\nu}{}_{\alpha\beta} g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta} g_{\alpha\nu} \tag{13}$$
$$-g_{\beta\mu,\alpha} = -\Gamma^{\nu}{}_{\beta\alpha} g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha} g_{\beta\nu} \tag{14}$$

We now add these three equations, and use the symmetry of the metric, so we have

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \left(\Gamma^{\nu}{}_{\alpha\mu} - \Gamma^{\nu}{}_{\mu\alpha}\right) g_{\beta\nu} + \left(\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\beta\alpha}\right) g_{\nu\mu} + \left(\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\nu}{}_{\mu\beta}\right) g_{\alpha\nu} \tag{15}$$

In the derivation for a coordinate basis, the first two terms are zero because of the symmetry of the Christoffel symbols. However, in a noncoordinate basis, this is no longer true. We now use 10 which, by comparison with 8 gives

$$\left(\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu}\right) \vec{e}_{\alpha} = c^{\alpha}{}_{\mu\nu} \vec{e}_{\alpha} \tag{16}$$

By comparing coefficients, we have

$$\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu} = c^{\alpha}{}_{\mu\nu} \tag{17}$$

Inserting this into 15 we have

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = c^{\nu}{}_{\mu\alpha} g_{\nu\beta} + c^{\nu}{}_{\beta\alpha} g_{\nu\mu} + \left(c^{\nu}{}_{\mu\beta} + 2\Gamma^{\nu}{}_{\mu\beta}\right) g_{\alpha\nu} \tag{18}$$

We now multiply both sides by $g^{\gamma\alpha}$ and use the fact that

$$g^{\gamma\alpha} g_{\alpha\nu} = \delta^{\gamma}{}_{\nu} \tag{19}$$

We have

$$g^{\gamma\alpha}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c^{\nu}{}_{\mu\alpha} g_{\nu\beta} - c^{\nu}{}_{\beta\alpha} g_{\nu\mu}\right) = \left(c^{\nu}{}_{\mu\beta} + 2\Gamma^{\nu}{}_{\mu\beta}\right) \delta^{\gamma}{}_{\nu} \tag{20}$$
$$= c^{\nu}{}_{\mu\beta} g_{\alpha\nu} g^{\gamma\alpha} + 2\Gamma^{\gamma}{}_{\mu\beta} \tag{21}$$

This gives the expression for the Christoffel symbols in terms of the metric and the coefficients $c^{\nu}{}_{\mu\alpha}$.

$$\Gamma^{\gamma}{}_{\mu\beta} = \frac{1}{2} g^{\gamma\alpha} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c^{\nu}{}_{\mu\alpha} g_{\nu\beta} - c^{\nu}{}_{\beta\alpha} g_{\nu\mu} - c^{\nu}{}_{\mu\beta} g_{\alpha\nu} \right) \tag{22}$$

The last term could be taken outside the parentheses, so we could write this as

$$\Gamma^{\gamma}{}_{\mu\beta} = \frac{1}{2} g^{\gamma\alpha} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c^{\nu}{}_{\mu\alpha} g_{\nu\beta} - c^{\nu}{}_{\beta\alpha} g_{\nu\mu} \right) - c^{\gamma}{}_{\mu\beta} \tag{23}$$