CHRISTOFFEL SYMBOLS IN TERMS OF THE METRIC TENSOR

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It's possible to calculate the Christoffel symbols from the metric tensor. We start with their definition in terms of the basis vectors in some coordinate system:

$$\boxed{\partial_j \mathbf{e}_i = \Gamma_{ij}^k \mathbf{e}_k} \tag{1}$$

We take the scalar product of this equation with another basis vector $\mathbf{e}_l$ and use the definition of the metric tensor as $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$:

$$\begin{aligned}
\Gamma_{ij}^k \mathbf{e}_k \cdot \mathbf{e}_l &= (\partial_j \mathbf{e}_i) \cdot \mathbf{e}_l && (2)\\
\Gamma_{ij}^k g_{kl} &= \partial_j (\mathbf{e}_i \cdot \mathbf{e}_l) - (\partial_j \mathbf{e}_l) \cdot \mathbf{e}_i && (3)\\
&= \partial_j g_{il} - \Gamma_{lj}^k \mathbf{e}_k \cdot \mathbf{e}_i && (4)\\
&= \partial_j g_{il} - \Gamma_{lj}^k g_{ki} && (5)\\
\Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki} &= \partial_j g_{il} && (6)
\end{aligned}$$

In this equation the index k is a dummy (being summed over), so only the indices i, j and l are specified. We can cyclically permute these indices to generate two more equations:

$$\begin{aligned}
\Gamma_{jl}^k g_{ki} + \Gamma_{il}^k g_{kj} &= \partial_l g_{ji} && (7)\\
\Gamma_{li}^k g_{kj} + \Gamma_{ji}^k g_{kl} &= \partial_i g_{lj} && (8)
\end{aligned}$$

We can now use the symmetry of the Christoffel symbols in the form

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{9}$$

to solve for Γ_{ij}^k by swapping indices in 7 and 8 to get

$$\begin{aligned}
\Gamma_{lj}^k g_{ki} + \Gamma_{il}^k g_{kj} &= \partial_l g_{ji} && (10)\\
\Gamma_{il}^k g_{kj} + \Gamma_{ij}^k g_{kl} &= \partial_i g_{lj} && (11)
\end{aligned}$$

We can now add 6 to 11 and subtract 10 to get

$$2\Gamma^k_{ij} g_{kl} = \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \tag{12}$$

Finally we can use the fact that

$$g^{ij} g_{jk} = \delta^i{}_k \tag{13}$$

and multiply both sides of 12 by g^{ml} to get

$$2\Gamma^k_{ij} g_{kl} g^{ml} = g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{14}$$

$$\Gamma^k_{ij} \delta^m_k = \frac{1}{2} g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{15}$$

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{16}$$

This gives us a formula for explicitly evaluating Christoffel symbols:

$$\boxed{\Gamma^m_{ij} = \frac{1}{2} g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right)} \tag{17}$$

This is a bit cumbersome to use as it requires finding the inverse metric tensor g^{ml} and has 3 sums over different derivatives.

As an example, we'll work out Γ^m_{ij} for 2-D polar coordinates. The metric tensor and its inverse here are:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{18}$$

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \tag{19}$$

so the derivatives are

$$\partial_r g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix} \tag{20}$$

$$\partial_\theta g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

Having only one non-zero derivative helps a lot, since the only non-zero term on the RHS of 17 is $\partial_r g_{\theta\theta} = 2r$. We'll work out a couple of the symbols explicitly and then give the final result.

$$\Gamma^{\theta}_{\ r\theta} = \frac{1}{2}g^{\theta l}\left(\partial_\theta g_{rl} + \partial_r g_{l\theta} - \partial_l g_{\theta r}\right) \tag{22}$$
$$= \frac{1}{2}g^{\theta\theta}\left(\partial_\theta g_{r\theta} + \partial_r g_{\theta\theta} - \partial_\theta g_{\theta r}\right) \tag{23}$$
$$= \frac{1}{2r^2}\left(2r\right) \tag{24}$$
$$= \frac{1}{r} \tag{25}$$
$$= \Gamma^{\theta}_{\ \theta r} \tag{26}$$
$$\Gamma^{r}_{\ \theta\theta} = \frac{1}{2}g^{rl}\left(\partial_\theta g_{\theta l} + \partial_\theta g_{l\theta} - \partial_l g_{\theta\theta}\right) \tag{27}$$
$$= \frac{1}{2}g^{rr}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \tag{28}$$
$$= \frac{1}{2}(-2r) \tag{29}$$
$$= -r \tag{30}$$

All of the other symbols are zero. The final results are

$$\Gamma^{r}_{\ ij} = \left[\begin{array}{cc} 0 & 0 \\ 0 & -r \end{array}\right] \tag{31}$$
$$\Gamma^{\theta}_{\ ij} = \left[\begin{array}{cc} 0 & r^{-1} \\ r^{-1} & 0 \end{array}\right] \tag{32}$$

Using these in 1 gives the 4 derivatives of the basis vectors:

$$\partial_r \mathbf{e}_r = \Gamma^{i}_{\ rr}\mathbf{e}_i = 0 \tag{33}$$
$$\partial_\theta \mathbf{e}_r = \Gamma^{i}_{\ r\theta}\mathbf{e}_i = \frac{1}{r}\mathbf{e}_\theta \tag{34}$$
$$\partial_r \mathbf{e}_\theta = \Gamma^{i}_{\ \theta r}\mathbf{e}_i = \frac{1}{r}\mathbf{e}_\theta \tag{35}$$
$$\partial_\theta \mathbf{e}_\theta = \Gamma^{i}_{\ \theta\theta}\mathbf{e}_i = -r\mathbf{e}_r \tag{36}$$

Note in particular that the middle two derivatives are the same, so that

$$\partial_\theta \mathbf{e}_r = \partial_r \mathbf{e}_\theta = \frac{1}{r}\mathbf{e}_\theta \tag{37}$$

This is an example of the symmetry of the Christoffel symbols in their lower two indices, so that $\Gamma^{i}_{\ r\theta} = \Gamma^{i}_{\ \theta r}$. This result depends on the basis vectors not necessarily being unit vectors, as in this case $\boldsymbol{e}_r$ is a unit vector but $\boldsymbol{e}_\theta$ is not:

$$\mathbf{e}_r = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{38}$$

$$\mathbf{e}_\theta = -r\sin\theta\hat{\mathbf{x}} + r\cos\theta\hat{\mathbf{y}} \tag{39}$$

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