CIRCULAR ORBITS - 3 MEASUREMENTS OF THE PERIOD

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We can generalize an earlier problem by working out the period of a circular orbit as measured by three different observers. The first observer is attached to the orbiting object which circles the mass at a radius r. The second observer is at rest at radius r, while the third observer is at infinity.

From the formula for angular momentum as measured by the orbiting object:

$$\ell^2 = \frac{r^2 GM}{r - 3GM} \tag{1}$$

we can calculate the angular speed from $\ell = r^2\omega$, so

$$\omega = \frac{\ell}{r^2} \tag{2}$$

$$= \frac{1}{r}\sqrt{\frac{GM}{r - 3GM}} \tag{3}$$

The period is then

$$T = \frac{2\pi}{\omega} \tag{4}$$

$$= 2\pi r\sqrt{\frac{r - 3GM}{GM}} \tag{5}$$

Defining the velocity as $2\pi r/T$, we get

$$v = \sqrt{\frac{GM}{r - 3GM}} \tag{6}$$

The period and velocity as measured at infinity are found from the angular speed at infinity Ω

$$\Omega = \frac{\sqrt{GM}}{r^{3/2}} \tag{7}$$

$$T_\infty = \frac{2\pi}{\Omega} \tag{8}$$

$$= 2\pi r\sqrt{\frac{r}{GM}} \tag{9}$$

$$v_\infty = \sqrt{\frac{GM}{r}} \tag{10}$$

Finally, the period T_0 as measured by the observer at rest at r is found from the relation between τ and t, as before.

$$\Delta\tau = \sqrt{1-\frac{2GM}{r}}\Delta t \tag{11}$$

In this case, $\Delta t = T_\infty$ and $\Delta\tau = T_0$ so

$$T_0 = 2\pi r\sqrt{1-\frac{2GM}{r}}\sqrt{\frac{r}{GM}} \tag{12}$$

$$= 2\pi r\sqrt{\frac{r-2GM}{GM}} \tag{13}$$

$$v_0 = \sqrt{\frac{GM}{r-2GM}} \tag{14}$$

Thus $T < T_0 < T_\infty$ and $v > v_0 > v_\infty$. The condition $T < T_0$ could be explained by time dilation, since to the observer at rest, the orbiting clock would run slow, so less time would appear to elapse on it than on the stationary clock. The condition $T_0 < T_\infty$ is a consequence of the difference between the proper time for an observer at rest at a finite distance from the mass and the time at infinity, which is given by the time coordinate t.

Since the square roots must all be real, we must have $r \ge 3GM$. This ensures that $v_\infty < v_0 \le 1$, but for $3GM < r < 4GM$, $v > 1$. I'm not entirely sure what the resolution of this apparent paradox is, but in the frame of the orbiting object, its own velocity is zero, so at best the expression for v above must be considered an artificial velocity which cannot be measured in any physical sense. If the speed of the object is measured by an external observer (such as the stationary observers at r and infinity) the value always seems to be less than 1.

COMMENTS

From Chris Kranenberg, 12 Nov 2022, 23:07 [Equation references are in Moore's book.]

There is an explanation for the the velocities having values greater than 1 for orbit radii between $3GM$ and $4GM$. Using the effective potential equation and expressing the angular momentum per unit mass in terms of orbit radius (Eq. 10.32 of P 10.7), for bound orbits the effective potential must be less than zero. Setting the effective potential equation to be less than zero and solving for the orbit radius gives $r > 4GM$. So the nonsense velocities forr_1 for the observer at rest. I believe your solution requires$r > 2GM$ to obtain real solutions and velocities less than 1 for$r > 3GM$, whereas my solution gives a range of radii, $3GM < r < 3.414$ for velocities greater than 1. This upper range measured in coordinate time is less than $4GM$ as measured in proper time due to the differences between coordinate time and proper time.

From Chris Kranenberg, 13 Nov 2022 18:11

There is a reason why some of the calculated velocities of the orbiting spaceship are greater than 1. Using Eq. 10.32 (the hint), insert into the effective potential equation to get Eq. 10.33 (see P 10.7 b). Circular orbits only exist for effective potential energies less than zero. The solution to 10.33 is radii greater than $4GM$. For orbital velocities greater than 1, the radii are less than $4GM$. So, assuming orbital radii exist less than $4GM$ gives contradictory solutions of orbital velocities greater than one. Therefore, any orbit radius less than $4GM$ will not exist. Since the effective potential energy is greater than one for radii less than $4GM$, any object spiraling in from infinity will return to infinity.

I calculated the stationary observer's measured angular velocity using $d\phi/d\tau$ (proper time measurement) times $d\tau/dt$ (from the Schwarzschild metric for a stationary observer). This result differs from the posted solution and yields a range of real radii, $3GM$ to $3.414GM$, where the measured velocity is greater than one. The posted solution yields real radii greater than $2GM$. But, it was shown in P 10.7 a that no circular orbits exist for radii less than $3GM$.

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