CIRCULAR ORBITS - KEPLER'S LAW

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We can now investigate some of the properties of an actual orbit in the Schwarzschild metric, that is, a path of motion that involves actually going around the central mass as opposed to moving only in the radial direction, which is what we've looked at so far.

The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right)=\frac{1}{2}\left(e^2-1\right) \tag{1}$$

We can write this in a more suggestive notation:

$$\tilde{K}+\tilde{V}(r)=\tilde{E} \tag{2}$$

where

$$\tilde{K}\equiv\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \tag{3}$$

$$\tilde{V}\equiv\frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right) \tag{4}$$

Although the radial coordinate isn't equivalent to the normal radial coordinate in flat space, it's sometimes useful to think of these two quantities as the radial kinetic and potential energies per unit mass, respectively.

We can show that the metric allows circular orbits as follows. For a circular orbit, the radial coordinate must be constant. (This isn't the same thing as centripetal acceleration, since that measures the magnitude of the change in the velocity *vector* as we move around the circular orbit. This vector is constantly changing direction throughout the orbit, while the magnitude of the radius always stays constant.)

This means that $\frac{dr}{d\tau}=0$, so $\tilde{K}=0$ and $\tilde{E}=\tilde{V}$. This condition, however, *could* apply at only a specific time, since the radial acceleration could be non-zero. For example, in an elliptical orbit, the radial velocity is zero momentarily when the object is at its closest and farthest approaches to the central mass. If we want to ensure that the acceleration is also zero (which would give us a circular orbit), we can take the derivative of 1 with respect to τ using the chain rule:

$$\frac{dr}{d\tau}\frac{d^2r}{d\tau^2}+\frac{d\tilde{V}}{dr}\frac{dr}{d\tau}=0 \tag{5}$$

$$\frac{d^2r}{d\tau^2}=-\frac{d\tilde{V}}{dr} \tag{6}$$

Thus the radial acceleration is zero if $\frac{d\tilde{V}}{dr}=0$. Using the definition of $\tilde{V}$ above, we get

$$\frac{d\tilde{V}}{dr}=GM\left(\frac{1}{r^2}+\frac{3\ell^2}{r^4}\right)-\frac{\ell^2}{r^3}=0 \tag{7}$$

Multiplying through by r^4 we get a quadratic equation:

$$GMr^2-\ell^2r+3GM\ell^2=0 \tag{8}$$

which has the solutions

$$r=\frac{\ell^2\pm\sqrt{\ell^4-12G^2M^2\ell^2}}{2GM} \tag{9}$$

We can convert this into another form by multiplying top and bottom by $\ell^2\mp\sqrt{\ell^4-12G^2M^2\ell^2}$

$$r=\frac{\ell^4-\left(\ell^4-12G^2M^2\ell^2\right)}{2GM\left(\ell^2\mp\sqrt{\ell^4-12G^2M^2\ell^2}\right)} \tag{10}$$

$$=\frac{6GM\ell^2}{\ell^2\mp\sqrt{\ell^4-12G^2M^2\ell^2}} \tag{11}$$

$$=\frac{6GM}{1\mp\sqrt{1-12\left(GM/\ell\right)^2}} \tag{12}$$

Thus there are two possible circular orbits for a given value of ℓ provided that the square root is real, which happens if

$$1-12\left(\frac{GM}{\ell}\right)^2\geq 0 \tag{13}$$

$$\ell\geq\sqrt{12}GM \tag{14}$$

It turns out that the inner orbit (with the + sign) is unstable and the outer orbit is stable.

As an example, suppose we have an object orbiting a neutron star with $GM = 2.2$ km. The object's angular momentum is $\ell = 6GM = 13.2$ km. The radius of the orbit is

$$r = \frac{6}{1-\sqrt{\frac{2}{3}}}GM \tag{15}$$

$$= 32.7GM \tag{16}$$

$$= 71.9 \text{ km} \tag{17}$$

We can get the angular speed from

$$\frac{d\phi}{d\tau} = \frac{\ell}{r^2} \tag{18}$$

$$= \frac{13.2}{(71.9)^2} \text{ km}^{-1} \tag{19}$$

$$= 0.00255 \text{ km}^{-1} \tag{20}$$

To convert this to non-relativistic units, we multiply by $c = 2.99 \times 10^5$ km s^{-1} and we have

$$\frac{d\phi}{d\tau} = 762.8 \text{ s}^{-1} \tag{21}$$

The orbital period as measured by the object is therefore

$$T = \frac{2\pi}{762.8} = 8.24 \text{ milliseconds} \tag{22}$$

For an observer at infinity (e.g. Earth) we need the angular velocity in the form $d\phi/dt$, since t is the time as measured by an observer at infinity. We can get this from the r component of the geodesic equation

$$\frac{d}{d\tau}\left(g_{rr}\frac{dr}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial r}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \tag{23}$$

For a circular orbit in the equatorial plane $dr/d\tau = d\theta/d\tau = 0$, so we get

$$\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2 + \partial_r g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{24}$$

Multiply through by $\left(\frac{d\tau}{dt}\right)^2$(remember that $\theta = \frac{\pi}{2}$):

$$\partial_r g_{tt} + \partial_r g_{\phi\phi}\left(\frac{d\phi}{d\tau}\frac{d\tau}{dt}\right)^2 = 0 \tag{25}$$

$$-\frac{2GM}{r^2} + 2r\left(\frac{d\phi}{dt}\right)^2 = 0 \tag{26}$$

In the last line, we used the Schwarzschild metric which states that

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2GM}{r}\right) \\ g_{\phi\phi} &= r^2 \sin\theta \end{aligned} \tag{27}$$

If we define $\Omega \equiv \frac{d\phi}{dt}$, we get

$$\Omega^2 = \frac{GM}{r^3} \tag{28}$$

or in terms of the period as measured at infinity

$$T_\infty = \frac{2\pi}{\Omega} \tag{29}$$

$$T_\infty^2 = \frac{4\pi^2 r^3}{GM} \tag{30}$$

This is the general relativistic form of Kepler's law of planetary orbits (the square of the period is proportional to the cube of the mean radius).

For the current example,

$$T_\infty = \frac{2\pi r^{3/2}}{\sqrt{GM}} \tag{31}$$

$$= \frac{2\pi\,(71.9)^{3/2}}{\sqrt{2.2}} \tag{32}$$

$$= 2582 \text{ km} \tag{33}$$

$$= 8.64 \text{ milliseconds} \tag{34}$$

From this we can work out the period T_0 as measured by an observer at rest at $r = 32.7GM$ by using the relation between t and τ:

$$\Delta\tau = \sqrt{1 - \frac{2GM}{r}}\Delta t \tag{35}$$

In this case, $\Delta t = T_\infty$ so

$$T_0 = \sqrt{1 - \frac{2}{32.7}}(8.64) \tag{36}$$
$$= 8.37 \text{ ms} \tag{37}$$

We thus get $T < T_0 < T_\infty$.

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