CIRCULAR ORBITS - SCHWARZSCHILD VS NEWTON

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 31 Aug 2022.

For a circular orbit, the Schwarzschild angular speed is obtained from the angular momentum:

$$\omega = \frac{d\phi}{d\tau} = \frac{\ell}{r^2} \tag{1}$$

$$= \frac{1}{r^2}\sqrt{\frac{r^2 GM}{r-3GM}} \tag{2}$$

We can write this as

$$\omega^2 = \frac{GM}{r^2\left(r-3GM\right)} \tag{3}$$

Comparing this with the angular speed measured at infinity $\Omega = \frac{d\phi}{dt}$ we have

$$\Omega^2 = \frac{GM}{r^3} \tag{4}$$

Thus $\omega > \Omega$.

The relation between r and ℓ for a circular orbit is

$$r = \frac{6GM}{1 \mp \sqrt{1-12\left(GM/\ell\right)^2}} \tag{5}$$

In the Newtonian case, we can use Kepler's law at all distances, so we have

$$\Omega^2 = \frac{\ell^2}{r^4} \tag{6}$$

$$= \frac{GM}{r^3} \tag{7}$$

$$r = \frac{\ell^2}{GM} \tag{8}$$

Note that we can get the same result by defining the Newtonian energy per unit mass

$$E_N = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_N \tag{9}$$

$$V_N = \frac{1}{2}\frac{\ell^2}{r^2} - \frac{GM}{r} \tag{10}$$

and then solving

$$\frac{dV_N}{dr} = -\frac{\ell^2}{r^3} + \frac{GM}{r^2} = 0 \tag{11}$$

In the Newtonian case, the potential energy V_N has only one minimum.

For large ℓ, we can expand the square root in 5 to get (using the minus sign)

$$r \to \frac{6GM}{1-\left(1-\frac{1}{2}\frac{12(GM)^2}{\ell^2}\right)} \tag{12}$$

$$= \frac{\ell^2}{GM} \tag{13}$$

Thus the Schwarzschild case reduces to the Newtonian case for large ℓ.