COMPONENTS OF ONE-FORMS AND VECTORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 24 Jan 2021.

A one-form is a tensor written $\binom{0}{1}$ which takes a single vector as input and produces a number as output. All tensors are entities that exist independently of any frame of reference, but the components of a tensor *do* depend on the reference frame. If a frame has as its basis vectors $\boldsymbol{e}_\alpha$, then the components of a tensor are found by inserting these basis vectors into the tensor's slots. We write the basis one-forms as $\boldsymbol{\omega}^\alpha$ where the α runs from 0 to 3, and denotes which basis one-form we're referring to, and *not* to the components of any one-form.

Thus the basis one-forms satisfy

$$\boldsymbol{\omega}^\alpha\left(\boldsymbol{e}_\beta\right)=\left\langle\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle=\delta_\beta^\alpha \tag{1}$$

A one-form can then be written in terms of its components and basis one-forms as

$$\boldsymbol{\sigma}=\sigma_\alpha\boldsymbol{\omega}^\alpha \tag{2}$$

Here, σ_α is the αth component of $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}^\alpha$ is the αth basis one-form. Bold faced symbols always refer to entire vectors or one-forms, while regular symbols with superscripts or subscripts refer to components.

Remembering that the expression

$$\langle\boldsymbol{\sigma},\boldsymbol{v}\rangle \tag{3}$$

is interpreted as the number of surfaces of the one-form $\boldsymbol{\sigma}$ that are pierced by the vector $\boldsymbol{v}$, we can find how many surfaces of some one-form $\boldsymbol{\sigma}$ are pierced by the basis vector $\boldsymbol{e}_\beta$:

$$\begin{aligned}
\left\langle\boldsymbol{\sigma},\boldsymbol{e}_\beta\right\rangle &= \left\langle\sigma_\alpha\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle && (4)\\
&= \sigma_\alpha\left\langle\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle && (5)\\
&= \sigma_\alpha\delta_\beta^\alpha && (6)\\
&= \sigma_\beta && (7)
\end{aligned}$$

The second line follows from the linearity of one-forms and the third line from 1. Thus the inner product $\left\langle \boldsymbol{\sigma}, \boldsymbol{e}_\beta \right\rangle$ of a one-form with a basis vector picks out the corresponding component of the one-form.

We can do a similar calculation to find the the number of surfaces of the basis one-form $\boldsymbol{\omega}^\alpha$ that are pierced by an arbitrary vector $\boldsymbol{v}$.

$$\langle \boldsymbol{\omega}^\alpha, \boldsymbol{v} \rangle = \left\langle \boldsymbol{\omega}^\alpha, v^\beta \boldsymbol{e}_\beta \right\rangle \tag{8}$$

$$= v^\beta \left\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \right\rangle \tag{9}$$

$$= v^\beta \delta^\alpha{}_\beta \tag{10}$$

$$= v^\alpha \tag{11}$$

Thus the inner product $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{v} \rangle$ of a vector with a basis one-form picks out the corresponding component of the vector.

Note that the components of one-forms have subscripts to label them, while the components of vectors have superscripts. If we combine 7 and 10 we find

$$\langle \boldsymbol{\sigma}, \boldsymbol{v} \rangle = \left\langle \sigma_\alpha \boldsymbol{\omega}^\alpha, v^\beta \boldsymbol{e}_\beta \right\rangle \tag{12}$$

$$= \sigma_\alpha v^\beta \left\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \right\rangle \tag{13}$$

$$= \sigma_\alpha v^\beta \delta^\alpha{}_\beta \tag{14}$$

$$= \sigma_\alpha v^\alpha \tag{15}$$

This is just another way of writing the scalar product of the vector corresponding to the one-form $\boldsymbol{\sigma}$ with the vector $\boldsymbol{v}$.

The scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be written in terms of the flat space metric $\eta_{\alpha\beta}$ as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u^\alpha v^\beta \eta_{\alpha\beta} \tag{16}$$

Since the one-form $\tilde{\boldsymbol{u}}$ corresponding to the vector $\boldsymbol{u}$ satisfies 15 (with $\boldsymbol{\sigma}$ replaced by $\boldsymbol{u}$), we have

$$u_\beta v^\beta = u^\alpha v^\beta \eta_{\alpha\beta} \tag{17}$$

Since this must be true for all vectors $\boldsymbol{v}$, we must have

$$u_\beta = u^\alpha \eta_{\alpha\beta} \tag{18}$$

The metric used in MTW is

$$\begin{aligned}\eta_{00} &= -1\\ \eta_{ii} &= +1\end{aligned} \tag{19}$$

with all other elements being zero. Therefore we can get the components of a one-form from the components of the corresponding vector:

$$\begin{aligned}u_0 &= -u^0\\ u_i &= u^i\end{aligned} \tag{20}$$

Since the metric $\eta_{\alpha\beta}$ multiplied by itself gives the unit matrix, $\eta_{\alpha\beta}$ is its own inverse. That is

$$\left\|\eta^{\alpha\beta}\right\| = \left\|\eta_{\alpha\beta}\right\|^{-1} = \left\|\eta_{\alpha\beta}\right\| \tag{21}$$

where the notation $\left\|\eta_{\alpha\beta}\right\|$ means 'the matrix with elements $\eta_{\alpha\beta}$. We can therefore multiply 18 by $\eta^{\beta\gamma}$ to get

$$u_\beta\eta^{\beta\gamma} = u^\alpha\eta_{\alpha\beta}\eta^{\beta\gamma} \tag{22}$$

$$= u^\alpha\delta^\gamma{}_\alpha \tag{23}$$

$$= u^\gamma \tag{24}$$

That is, we can use the inverse metric to raise an index to recover a vector's component from the components of its corresponding one-form.

Finally, we see there are several ways to write the scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. One way is given in 15:

$$\boldsymbol{u}\cdot\boldsymbol{v} = \langle\boldsymbol{u},\boldsymbol{v}\rangle = u_\alpha v^\alpha \tag{25}$$

Using 18 and 24 we can raise or lower the index on one of the vector components to get

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_\alpha v_\beta\eta^{\alpha\beta} = u^\alpha v^\beta\eta_{\alpha\beta} \tag{26}$$

PINGBACKS

Pingback: Gradient as a one-form