CONSERVED QUANTITIES FROM KILLING VECTORS

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A Killing vector is a vector that lies along a direction in which the metric doesn't change, that is, the metric $g_{\mu\nu}$ is independent of the coordinate in a given direction. A Killing vector can be used to discover a conserved quantity in a given metric. To see this, we start with the Lagrangian for a given metric:

$$L=\sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}} \tag{1}$$

where σ is a parameter along an object's world line and the x^{μ} are the coordinates. The proper time along a world line is obtained from the integral

$$\Delta\tau=\int_{0}^{1}\sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}}d\sigma=\int_{0}^{1}L\;d\sigma \tag{2}$$

Here we've chosen the parameter σ so that the world line starts at $\sigma=0$ and ends at $\sigma=1$. From this relation, we see that

$$L=\frac{d\tau}{d\sigma} \tag{3}$$

From the Lagrangian we obtain the Euler-Lagrange equations

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^{\mu}/d\sigma\right)}\right)=\frac{\partial L}{\partial x^{\mu}} \tag{4}$$

From 1 we see that if the metric $g_{\mu\nu}$ is independent of a particular coordinate x^{μ}, then $\partial L/\partial x^{\mu}=0$ and

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^{\mu}/d\sigma\right)}\right)=0 \tag{5}$$

so that $\frac{\partial L}{\partial(dx^{\mu}/d\sigma)}$ is a constant, and is therefore conserved.

Because the Lagrangian in general relativity always has the form 1, we can rewrite it as follows, assuming that the metric does not depend on any of the derivatives $dx^{\mu}/d\sigma$.

$$\frac{\partial L}{\partial\left(dx^{\alpha}/d\sigma\right)} = \frac{1}{2}\frac{1}{\sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}}}\left(-2g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma}\right) \tag{6}$$

$$= -\frac{1}{L}g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma} \tag{7}$$

$$= -\frac{d\sigma}{d\tau}g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma} \tag{8}$$

$$= -g_{\alpha\nu}\frac{dx^{\nu}}{d\tau} \tag{9}$$

The factor of 2 in the first line comes from the fact that

$$\frac{d}{d\left(dx^{\alpha}/d\sigma\right)}\left[g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}\right] = g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma}+g_{\mu\alpha}\frac{dx^{\mu}}{d\sigma} \tag{10}$$

$$= g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma}+g_{\nu\alpha}\frac{dx^{\nu}}{d\sigma} \tag{11}$$

$$= g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma}+g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma} \tag{12}$$

$$= 2g_{\alpha\nu}\frac{dx^{\nu}}{d\sigma} \tag{13}$$

In the second line, we relabel the summed index $\mu\to\nu$. In the third line, we use the symmetry of the metric $g_{\alpha\nu}=g_{\nu\alpha}$.

Suppose that the metric is independent of the coordinate x^{1}. Then a corresponding Killing vector is

$$\boldsymbol{\xi} = (0,1,0,0) \tag{14}$$

For $\mu=1$ we can write 9 as (using 5)

$$-g_{\alpha\nu}\frac{dx^{\nu}}{d\tau} = -g_{1\nu}\frac{dx^{\nu}}{d\tau} = -g_{\mu\nu}\xi^{\mu}\frac{dx^{\nu}}{d\tau} = -g_{\mu\nu}\xi^{\mu}u^{\nu} = -\boldsymbol{\xi}\cdot\boldsymbol{u} = \text{constant} \tag{15}$$

where u^{ν} is the object's four-velocity $u^{\nu}=dx^{\nu}/d\tau$.

In vector notation, we have the condition

$$\boxed{\boldsymbol{\xi}\cdot\boldsymbol{u} = \text{constant}} \tag{16}$$

Since 15 is a tensor equation, it is valid in all coordinate systems. As $\boldsymbol{u}$ is proportional to the object's four-momentum, this condition can also be written as

$$\boxed{\boldsymbol{\xi}\cdot\boldsymbol{p} = \text{constant}} \tag{17}$$

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