COVARIANT DERIVATIVE IN SEMI-LOG COORDINATES

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As another example of using the geodesic equation to calculate Christoffel symbols, we'll consider the semi-log coordinates introduced earlier:

$$p = x \tag{1}$$

$$q = e^{by} \tag{2}$$

The invariant interval is

$$ds^2 = dx^2 + dy^2 \tag{3}$$

$$= dp^2 + \frac{1}{(bq)^2} dq^2 \tag{4}$$

so the metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(bq)^2} \end{bmatrix} \tag{5}$$

We compare the geodesic equation:

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j \dot{x}^i = 0 \tag{6}$$

with the expression for the Christoffel symbols:

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j \dot{x}^i = 0 \tag{7}$$

We get

$$\ddot{p} = 0 \tag{8}$$

$$\frac{1}{(bq)^2}\ddot{q} - \frac{2}{b^2 q^3}\dot{q}^2 - \frac{1}{2}\left(-\frac{2}{b^2 q^3}\dot{q}^2\right) = 0 \tag{9}$$

$$\ddot{q} - \frac{1}{q}\dot{q}^2 = 0 \tag{10}$$

The Christoffel symbols are thus

$$\Gamma^{p}{}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

$$\Gamma^{q}{}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{q} \end{bmatrix} \tag{12}$$

Now consider a vector field $A^i = [0, Cx]$ (where C is a constant) in rectangular coordinates. In these coordinates, its covariant derivative is just the normal derivative, so

$$\nabla_i A^j = \partial_i A^j \tag{13}$$

$$= \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \tag{14}$$

In the semi-log system, we have

$$\nabla_i A^j = \partial_i A^j + \Gamma^{j}{}_{ik} A^k \tag{15}$$

To use this formula we need A^i in the pq system, which we can get by the usual transformation, using:

$$A^p = A^i \partial_i p = A^x = 0 \tag{16}$$

$$A^q = A^i \partial_i q = A^y \left(b e^{by} \right) = Cbpq \tag{17}$$

With these transformations, we get

$$\nabla_i A^j = \begin{bmatrix} 0 & Cbq \\ 0 & Cbp - Cbp \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 0 & Cbq \\ 0 & 0 \end{bmatrix} \tag{19}$$

The difference in gradients is because of the different scales used in the vertical direction. Consider first the vertical component of A. In rectangular coordinates this has a constant value for a given value of x namely $A^y = Cx$. However, if we use q to measure vertical distance, a unit change in y results in a larger and larger change in q the higher up the vertical axis we go, so A^q must increase as q increases, even if $x = p$ is held constant. This is reflected by 17, where A^q is proportional to q as well as p.

If we used ordinary derivatives to calculate the gradient of A in the pq system, we would get $\partial_q A^q = Cbp$. However, the 'true' value of the vertical component of A doesn't change as we move up or down a vertical line, and this is reflected by the $\Gamma^{q}{}_{ik} A^k = -Cbp$ correction term that is present in the covariant derivative 15, with the result that $\nabla_q A^q = \partial_q A^q + \Gamma^q{}_{qk} A^k = Cbp - Cbp = 0$.

The value of $\nabla_p A^q = Cbq$ again reflects the fact that a vertical component that is constant in rectangular coordinates must get numerically larger with increasing height in the pq system.

As a final test that all is well, we can use the standard tensor transformation to transform 19 back to rectangular coordinates, where we denote rectangular coordinates by r^i and the semi-log pq system by s:

$$\left[\nabla_i A^j\right]_r = \left[\nabla_a A^b\right]_s \frac{\partial s^a}{\partial r^i} \frac{\partial r^j}{\partial s^b} \tag{20}$$

The only non-zero component of $\left[\nabla_a A^b\right]_s$ is $\left[\nabla_p A^q\right]_s = Cbq$, so the RHS has only one non-zero term, which occurs when $a = p$, $i = x$, $b = q$ and $j = y$. For this term we have

$$\begin{aligned} \left[\nabla_x A^y\right]_r &= Cbq \frac{\partial p}{\partial x} \frac{\partial y}{\partial q} && (21) \\ &= Cbq\,(1)\,\frac{1}{bq} && (22) \\ &= C && (23) \end{aligned}$$

The overall transformation thus gives 14 back again.