COVARIANT DERIVATIVE IS A TENSOR

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The Christoffel symbols appear when we take the total or covariant derivative of a tensor in a general coordinate system. They arise because, in general, the derivatives of the basis vectors in a general coordinate system are not zero. They are defined as

$$\frac{\partial \boldsymbol{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma{}_{\alpha\beta}\boldsymbol{e}_\gamma \tag{1}$$

where the $\boldsymbol{e}_\gamma$ are the basis vectors. That is, they are the components of the derivatives of the basis vectors (which are vectors in their own right) in terms of the same basis vectors.

The total derivative of a vector $\vec{V}$ is then

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\gamma \Gamma^\alpha{}_{\gamma\beta} \tag{2}$$

Note the use of a semicolon on the LHS and a comma on the RHS to denote, respectively, the covariant derivative and the ordinary derivative with respect to the coordinate x^β.

It turns out that each of the two terms on the RHS of 2 is not itself a tensor, although their sum is. Seeing this is a bit tricky. To do it, we need to examine how 2 transforms under a change of coordinates. We'll work through this for the transformation of a vector; the proof for a one-form is similar.

First, recall that under a change of coordinates, a vector transforms as (primed indexes refer to the new coordinate system; unprimed to the old):

$$V^{\sigma'} = \Lambda^{\sigma'}{}_\alpha V^\alpha \tag{3}$$

where $\Lambda^{\sigma'}{}_\alpha$ is the matrix of derivatives

$$\left[\Lambda^{\sigma'}{}_\alpha\right] = \left[\frac{\partial x^{\sigma'}}{\partial x^\alpha}\right] \tag{4}$$

For changing from rectangular to polar, for example, we have

$$\Lambda^{\sigma'}{}_{\alpha} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \tag{5}$$

The basis vectors transform oppositely to the vector components, so we have

$$\boldsymbol{e}_{\sigma'} = \Lambda^{\alpha}{}_{\sigma'}\boldsymbol{e}_{\alpha} \tag{6}$$

where

$$[\Lambda^{\alpha}{}_{\sigma'}] = \left[\frac{\partial x^{\alpha}}{\partial x^{\sigma'}}\right] \tag{7}$$

Now consider how 2 transforms. We'll look at the first term $V^{\alpha}{}_{,\beta}$. We transform the vector component first to get

$$V^{\nu'}{}_{,\mu'} = \left(\Lambda^{\nu'}{}_{\beta}V^{\beta}\right)_{,\mu'} \tag{8}$$

Now we apply the product rule to get

$$V^{\nu'}{}_{,\mu'} = \left(\Lambda^{\nu'}{}_{\beta}\right)_{,\mu'} V^{\beta} + \Lambda^{\nu'}{}_{\beta}V^{\beta}{}_{,\mu'} \tag{9}$$

The second term can be expanded using the chain rule.

$$V^{\beta}{}_{,\mu'} = V^{\beta}{}_{,\alpha}\Lambda^{\alpha}{}_{\mu} \tag{10}$$

Thus we have

$$V^{\nu'}{}_{,\mu'} = \left(\Lambda^{\nu'}{}_{\beta}\right)_{,\mu'} V^{\beta} + \Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}V^{\beta}{}_{,\alpha} \tag{11}$$

This shows that the ordinary derivative $V^{\alpha}{}_{,\beta}$ does not transform like a tensor. If it did, the transformation would be

$$V^{\nu'}{}_{,\mu'} = \Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}V^{\beta}{}_{,\alpha} \tag{12}$$

That is, we'd have one Λ matrix for each of the two indexes. However, the first term on the RHS of 11 is in general not zero, so the ordinary derivative $V^{\alpha}{}_{,\beta}$ does not transform like a tensor.

Now consider the second term on the RHS of 2. We wish to find

$$V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'} \tag{13}$$

To do this, we use 1, which shows that $\Gamma^{\nu'}{}_{\sigma'\mu'}$ is the ν' component of $\frac{\partial \boldsymbol{e}_{\sigma'}}{\partial x^{\mu'}}$. So we have

$$V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'} = V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\sigma'}}{\partial x^{\mu'}}\right]^{\nu'} \tag{14}$$

We now use 6 to write the basis vector $\boldsymbol{e}_{\sigma'}$ in terms of the basis vectors from the original coordinates.

$$V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\sigma'}}{\partial x^{\mu'}}\right]^{\nu'} = V^{\sigma'}\left[\frac{\partial\left(\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma'}\right)}{\partial x^{\mu'}}\right]^{\nu'} \tag{15}$$

The term in brackets is the ν' component of a vector in the new coordinates, so we can write it in terms of the old coordinates using 4.

$$V^{\sigma'}\left[\frac{\partial\left(\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma'}\right)}{\partial x^{\mu'}}\right]^{\nu'} = V^{\sigma'}\left[\frac{\partial\left(\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma'}\right)}{\partial x^{\mu'}}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} \tag{16}$$

We now expand the derivative using the product rule.

$$V^{\sigma'}\left[\frac{\partial\left(\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma'}\right)}{\partial x^{\mu'}}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} = V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\mu'}}\Lambda^{\omega}{}_{\sigma'} + \boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} \tag{17}$$

We now express the first derivative in terms of the old coordinates.

$$V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\mu'}}\Lambda^{\omega}{}_{\sigma'} + \boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} = V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\alpha}}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\omega}{}_{\sigma'} + \boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} \tag{18}$$

$$= V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\alpha}}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\omega}{}_{\sigma'} + \boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta}\Lambda^{\nu'}{}_{\beta} \tag{19}$$

In the second term in brackets, the β index refers to the component of $V^{\sigma'}\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}$ that lies along the basis vector $\boldsymbol{e}_{\beta}$ (since both β and ω are unprimed, they refer to the same coordinate system). However, from the definition of basis vectors, the β component is nonzero only along the $\boldsymbol{e}_{\beta}$ basis vector, so

$$\left[\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta} = \delta^{\beta}{}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'} \tag{20}$$

Also, from the definition 1 of the Christoffel symbols, the β component of $\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\alpha}}$ is $\Gamma^{\beta}{}_{\omega\alpha}$, so we have

$$V^{\sigma'}\left[\frac{\partial \boldsymbol{e}_{\omega}}{\partial x^{\alpha}}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\omega}{}_{\sigma'}+\boldsymbol{e}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]^{\beta}\Lambda^{\nu'}{}_{\beta}=V^{\sigma'}\left[\Gamma^{\beta}{}_{\omega\alpha}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\omega}{}_{\sigma'}\Lambda^{\nu'}{}_{\beta}+\delta^{\beta}{}_{\omega}\Lambda^{\omega}{}_{\sigma',\mu'}\right]\Lambda^{\nu'}{}_{\beta} \quad (21)$$

$$V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'}=V^{\omega}\Gamma^{\beta}{}_{\omega\alpha}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}V^{\sigma'} \quad (22)$$

where we've inserted the original term 13 on the LHS in the last line. We thus see that this term $V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'}$ does not transform as a tensor due to the extra term on the RHS of 22.

What we'd like to show, however, is that the *sum* $V^{\alpha}{}_{,\beta}+V^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta}$ *does* transform like a tensor. Adding 11 and 22 we have

$$\begin{aligned}V^{\nu'}{}_{,\mu'}+V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'} &= \left(\Lambda^{\nu'}{}_{\beta}\right)_{,\mu'}V^{\beta}+\Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}V^{\beta}{}_{,\alpha}+\\ &\quad V^{\omega}\Gamma^{\beta}{}_{\omega\alpha}\Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}V^{\sigma'} \quad (23)\\ &= \left(V^{\beta}{}_{,\alpha}+V^{\omega}\Gamma^{\beta}{}_{\omega\alpha}\right)\Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta}+\\ &\quad \Lambda^{\nu'}{}_{\beta,\mu'}V^{\beta}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}V^{\sigma'} \quad (24)\end{aligned}$$

We need to show that the last two terms sum up to zero. To do this, we use $V^{\beta}=\Lambda^{\beta}{}_{\sigma'}V^{\sigma'}$ to get

$$\Lambda^{\nu'}{}_{\beta,\mu'}V^{\beta}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}V^{\sigma'}=\left[\Lambda^{\nu'}{}_{\beta,\mu'}\Lambda^{\beta}{}_{\sigma'}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}\right]V^{\sigma'} \quad (25)$$

The terms in brackets are the expansion of a derivative using the product rule. That is

$$\Lambda^{\nu'}{}_{\beta,\mu'}\Lambda^{\beta}{}_{\sigma'}+\Lambda^{\beta}{}_{\sigma',\mu'}\Lambda^{\nu'}{}_{\beta}=\left[\Lambda^{\nu'}{}_{\beta}\Lambda^{\beta}{}_{\sigma'}\right]_{,\mu'} \quad (26)$$

However, the matrices $\Lambda^{\beta}{}_{\sigma'}$ and $\Lambda^{\nu'}{}_{\beta}$ are inverses of each other, so their product is the identity matrix, which is a constant, so its derivative is zero. Thus we get our final result

$$V^{\nu'}{}_{,\mu'}+V^{\sigma'}\Gamma^{\nu'}{}_{\sigma'\mu'}=\left(V^{\beta}{}_{,\alpha}+V^{\omega}\Gamma^{\beta}{}_{\omega\alpha}\right)\Lambda^{\alpha}{}_{\mu'}\Lambda^{\nu'}{}_{\beta} \quad (27)$$

which shows that that the covariant derivative $V^{\beta}{}_{;\alpha}$ does indeed transform as a tensor.

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