

COVARIANT DERIVATIVE OF A GENERAL TENSOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 29 Jan 2021.

We've seen how to define the covariant derivative or absolute gradient of a contravariant vector, giving the formula:

$$\nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i \Gamma_{ij}^k \quad (1)$$

The covariant derivative of this vector is a tensor, unlike the ordinary derivative. Here we see how to generalize this to get the absolute gradient of tensors of any rank.

First, let's find the covariant derivative of a covariant vector (one-form) B_i . The starting point is to consider $\nabla_j (A^i B_i)$. The quantity $A^i B_i$ is a scalar, and to proceed we *require* two conditions:

- (1) The covariant derivative of a scalar is the same as the ordinary derivative.
- (2) The covariant derivative obeys the product rule.

These two conditions aren't derived; they are just required as part of the definition of the covariant derivative.

Using rule 2, we have

$$\nabla_j (A^i B_i) = (\nabla_j A^i) B_i + A^i (\nabla_j B_i) \quad (2)$$

$$= \left(\partial_j A^i + A^k \Gamma_{kj}^i \right) B_i + A^i (\nabla_j B_i) \quad (3)$$

We now apply rule 1 to the LHS:

$$\nabla_j (A^i B_i) = \partial_j (A^i B_i) \quad (4)$$

$$= (\partial_j A^i) B_i + A^i (\partial_j B_i) \quad (5)$$

Equating 3 and 5 we get

$$\left(\partial_j A^i + A^k \Gamma_{kj}^i\right) B_i + A^i (\nabla_j B_i) = (\partial_j A^i) B_i + A^i (\partial_j B_i) \quad (6)$$

$$B_i A^k \Gamma_{kj}^i + A^i (\nabla_j B_i) = A^i (\partial_j B_i) \quad (7)$$

$$B_i A^k \Gamma_{kj}^i + A^k (\nabla_j B_k) = A^k (\partial_j B_k) \quad (8)$$

$$[B_i \Gamma_{kj}^i + (\nabla_j B_k)] A^k = A^k (\partial_j B_k) \quad (9)$$

In 8 we've relabelled the dummy index i to k in the second and third terms so we could factor out A^k in the last line. Since the vector A^k is arbitrary, the factors multiplying it on each side must be equal, so we get

$$B_i \Gamma_{kj}^i + (\nabla_j B_k) = \partial_j B_k \quad (10)$$

from which we can get the covariant derivative of B_k :

$$\boxed{\nabla_j B_k = \partial_j B_k - B_i \Gamma_{kj}^i} \quad (11)$$

To extend this argument to a tensor of higher rank with mixed indices, we generalize this argument. First, we contract all the indices of the tensor with covariant or contravariant vectors, as appropriate, to produce a scalar, and then apply the product rule to the scalar. So for example

$$\begin{aligned} \nabla_l \left(C_{jk}^i A_i B^j D^k \right) &= \nabla_l \left(C_{jk}^i \right) A_i B^j D^k + C_{jk}^i \nabla_l \left(A_i \right) B^j D^k + \\ &C_{jk}^i A_i \nabla_l \left(B^j \right) D^k + C_{jk}^i A_i B^j \nabla_l \left(D^k \right) \end{aligned} \quad (12)$$

$$\begin{aligned} &= \partial_l \left(C_{jk}^i \right) A_i B^j D^k + C_{jk}^i \partial_l \left(A_i \right) B^j D^k + \\ &C_{jk}^i A_i \partial_l \left(B^j \right) D^k + C_{jk}^i A_i B^j \partial_l \left(D^k \right) \end{aligned} \quad (13)$$

We now substitute for the covariant derivatives of the vectors in 12 and set the result equal to 13, and cancel terms. We then use the fact that A_i , B^j and D^k are all arbitrary so the factors on each side of the equation multiplying them must be equal. The result is

$$\nabla_l C_{jk}^i = \partial_l C_{jk}^i + \Gamma_{lm}^i C_{jk}^m - \Gamma_{lj}^m C_{mk}^i - \Gamma_{lk}^m C_{jm}^i \quad (14)$$

In general, the rule is that for each contravariant (upper) index in the tensor, there is a positive term with a Christoffel symbol, and for each covariant (lower) index, there is a negative term.

PINGBACKS

- Pingback: Christoffel symbols - symmetry
- Pingback: Stress-energy tensor - conservation equations
- Pingback: Derivatives of a one-form field in polar coordinates
- Pingback: Maxwell's equations in cylindrical coordinates
- Pingback: Covariant derivative of the metric tensor
- Pingback: Covariant derivative - commutativity
- Pingback: Black hole with static charge; Reissner-Nordström solution
- Pingback: Einstein equation solution for the interior of a spherically symmetric star