COVARIANT DERIVATIVE OF A GENERAL TENSOR

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We've seen how to define the covariant derivative or absolute gradient of a contravariant vector, giving the formula:

$$\boxed{\nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i \Gamma_{ij}^k} \tag{1}$$

The covariant derivative of this vector is a tensor, unlike the ordinary derivative. Here we see how to generalize this to get the absolute gradient of tensors of any rank.

First, let's find the covariant derivative of a covariant vector (one-form) B_i. The starting point is to consider $\nabla_j \left(A^i B_i\right)$. The quantity $A^i B_i$ is a scalar, and to proceed we *require* two conditions:

(1) The covariant derivative of a scalar is the same as the ordinary derivative.
(2) The covariant derivative obeys the product rule.

These two conditions aren't derived; they are just required as part of the definition of the covariant derivative.

Using rule 2, we have

$$\nabla_j \left(A^i B_i\right) = \left(\nabla_j A^i\right) B_i + A^i \left(\nabla_j B_i\right) \tag{2}$$

$$= \left(\partial_j A^i + A^k \Gamma_{kj}^i\right) B_i + A^i \left(\nabla_j B_i\right) \tag{3}$$

We now apply rule 1 to the LHS:

$$\nabla_j \left(A^i B_i\right) = \partial_j \left(A^i B_i\right) \tag{4}$$

$$= \left(\partial_j A^i\right) B_i + A^i \left(\partial_j B_i\right) \tag{5}$$

Equating 3 and 5 we get

$$\left(\partial_j A^i + A^k \Gamma^i_{\ kj}\right) B_i + A^i \left(\nabla_j B_i\right) = \left(\partial_j A^i\right) B_i + A^i \left(\partial_j B_i\right) \tag{6}$$

$$B_i A^k \Gamma^i_{\ kj} + A^i \left(\nabla_j B_i\right) = A^i \left(\partial_j B_i\right) \tag{7}$$

$$B_i A^k \Gamma^i_{\ kj} + A^k \left(\nabla_j B_k\right) = A^k \left(\partial_j B_k\right) \tag{8}$$

$$\left[B_i \Gamma^i_{\ kj} + \left(\nabla_j B_k\right)\right] A^k = A^k \left(\partial_j B_k\right) \tag{9}$$

In 8 we've relabelled the dummy index i to k in the second and third terms so we could factor out A^k in the last line. Since the vector A^k is arbitrary, the factors multiplying it on each side must be equal, so we get

$$B_i \Gamma^i_{\ kj} + \left(\nabla_j B_k\right) = \partial_j B_k \tag{10}$$

from which we can get the covariant derivative of B_k:

$$\boxed{\nabla_j B_k = \partial_j B_k - B_i \Gamma^i_{\ kj}} \tag{11}$$

To extend this argument to a tensor of higher rank with mixed indices, we generalize this argument. First, we contract all the indices of the tensor with covariant or contravariant vectors, as appropriate, to produce a scalar, and then apply the product rule to the scalar. So for example

$$\begin{aligned} \nabla_l \left(C^i_{jk} A_i B^j D^k\right) &= \nabla_l \left(C^i_{jk}\right) A_i B^j D^k + C^i_{jk} \nabla_l \left(A_i\right) B^j D^k + \\ &\quad C^i_{jk} A_i \nabla_l \left(B^j\right) D^k + C^i_{jk} A_i B^j \nabla_l \left(D^k\right) \qquad (12) \\ &= \partial_l \left(C^i_{jk}\right) A_i B^j D^k + C^i_{jk} \partial_l \left(A_i\right) B^j D^k + \\ &\quad C^i_{jk} A_i \partial_l \left(B^j\right) D^k + C^i_{jk} A_i B^j \partial_l \left(D^k\right) \qquad (13) \end{aligned}$$

We now substitute for the covariant derivatives of the vectors in 12 and set the result equal to 13, and cancel terms. We then use the fact that A_i, B^j and D^k are all arbitrary so the factors on each side of the equation multiplying them must be equal. The result is

$$\nabla_l C^i_{jk} = \partial_l C^i_{jk} + \Gamma^i_{lm} C^m_{jk} - \Gamma^m_{lj} C^i_{mk} - \Gamma^m_{lk} C^i_{jm} \tag{14}$$

In general, the rule is that for each contravariant (upper) index in the tensor, there is a positive term with a Christoffel symbol, and for each covariant (lower) index, there is a negative term.

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