COVARIANT DERIVATIVE OF A VECTOR IN THE SCHWARZSCHILD METRIC

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Post date: 20 Jan 2023.

Here's another example of calculating the covariant derivative in the Schwarzschild (S) metric. We're given a vector with coordinates in the S metric of:

$$\mathbf{v}=\left[1-\frac{2GM}{r},0,0,0\right] \tag{1}$$

The covariant derivative is given by

$$\nabla_j v^k \equiv \frac{\partial v^k}{\partial x^j}+v^i\Gamma^k_{ij} \tag{2}$$

Since the only non-zero component of **v** is v^t and it depends only on r, most of the terms are zero.

The Christoffel symbols for the S metric are:

$$\Gamma^t_{ij}=\begin{bmatrix} 0 & \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$\Gamma^r_{ij}=\begin{bmatrix} \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\ 0 & -\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{2GM}{r}\right) \end{bmatrix} \tag{4}$$

$$\Gamma^\theta_{ij}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{5}$$

$$\Gamma^\phi_{ij}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \cot\theta\\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \tag{6}$$

The one non-zero derivative is

$$\frac{\partial v^t}{\partial r} = \frac{2GM}{r^2} \tag{7}$$

and the values of the second term in 2 are

$$v^i \Gamma^t_{\ ir} = \frac{GM}{r^2} \tag{8}$$

$$v^i \Gamma^r_{\ it} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^2 \tag{9}$$

with all other terms being zero.

The covariant derivative is then (with j the row index and k the column index):

$$\nabla_j v^k = \begin{bmatrix} 0 & \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^2 & 0 & 0 \\ \frac{3GM}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$