CURVATURE OF 2-DIMENSIONAL SPACE

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Consider the metric

$$ds^2 = dr^2 + (rh(r))^2 d\theta^2 \tag{1}$$

For $h(r) = 1$, this is the ordinary 2-dimensional metric in polar coordinates. We'll use the method of comparing the circumference of a small circle to its value in flat space to investigate the curvature if $h(r) \neq 1$.

Starting at the origin, the radius of a small circle out to a distance $r = \epsilon$ is given by

$$\text{radius} = \int_0^{\epsilon} dr = \epsilon \tag{2}$$

The circumference of this circle is then given by

$$\text{circum} = \int_0^{2\pi} \epsilon h(\epsilon) d\theta = 2\pi\epsilon h(\epsilon) \tag{3}$$

The formula for the curvature is

$$R = \lim_{\text{radius}\to 0} \frac{6}{\text{radius}^2}\left(1 - \frac{\text{circumference}}{2\pi\text{radius}}\right) \tag{4}$$

In our case, we have

$$R = \lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{2\pi\epsilon h(\epsilon)}{2\pi\epsilon}\right) \tag{5}$$

$$= \lim_{\epsilon\to 0} \frac{6}{\epsilon^2}(1 - h(\epsilon)) \tag{6}$$

If $h(0) = 1$, we can consider what happens if $h(r)$ increases or decreases from this point. If $h(r)$ increases from 1, then R becomes negative; if it decreases then R becomes positive.

For the case

$$h(r) = \frac{\sin r}{r} \tag{7}$$

we can expand it in a series about $r = 0$ to get

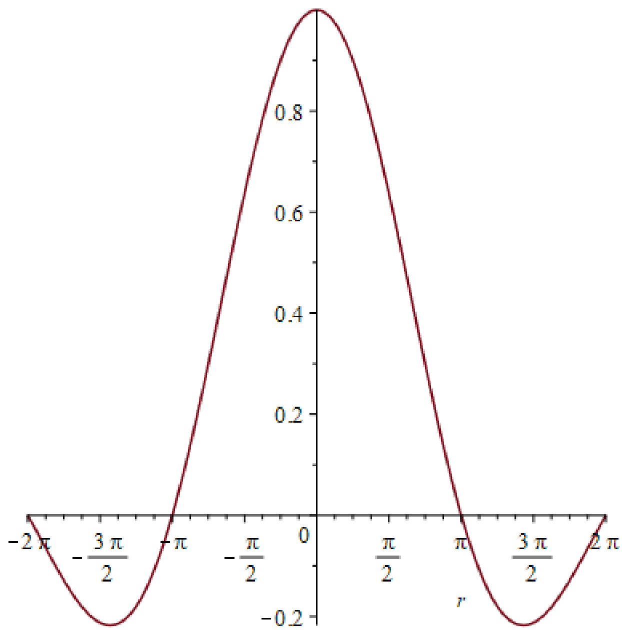

FIGURE 1. Graph of $\frac{\sin r}{r}$.

$$h(\epsilon) = \frac{1}{\epsilon}\left(\epsilon - \frac{\epsilon^3}{3!} + \ldots\right) \tag{8}$$

$$= 1 - \frac{\epsilon^2}{6} + \ldots \tag{9}$$

so the curvature becomes

$$R = \lim_{\epsilon\to 0}\frac{6}{\epsilon^2}\left(\frac{\epsilon^2}{6} + \ldots\right) \tag{10}$$

$$= +1 \tag{11}$$

The graph of $\frac{\sin r}{r}$ is shown in Fig. 1, where we see that $h(r)$ decreases from its value at $r = 0$, which indicates a positive curvature.

For

$$h(r) = \frac{\sinh r}{r} \tag{12}$$

the series is

$$h(\epsilon) = \frac{1}{\epsilon}\left(\epsilon + \frac{\epsilon^3}{3!} + \ldots\right) \tag{13}$$

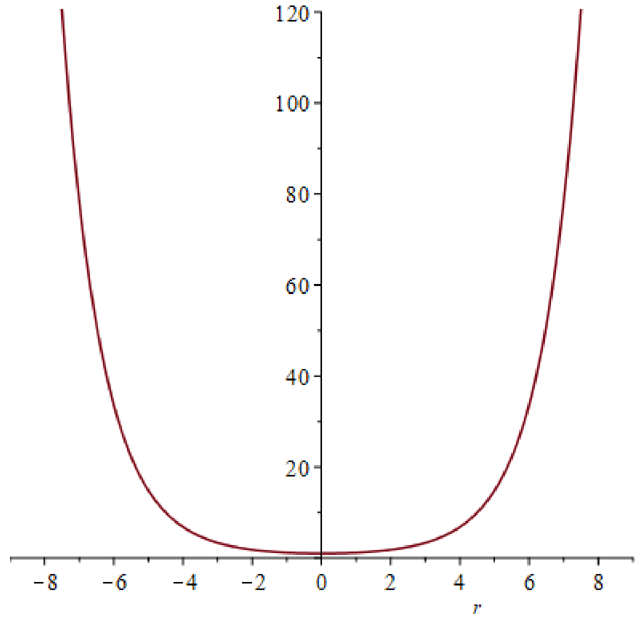

FIGURE 2. Graph of $\frac{\sinh r}{r}$.

so the curvature becomes

$$R = \lim_{\epsilon \to 0} \frac{6}{\epsilon^2} \left(-\frac{\epsilon^2}{6} + \dots \right) \tag{14}$$

$$= -1 \tag{15}$$

The graph of $\frac{\sinh r}{r}$ is shown in Fig. 2, where we see that $h(r)$ increases from its value at $r = 0$, indicating a negative curvature.