

CURVATURE OF A SPHERE

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The metric on a unit sphere is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (1)$$

We can calculate the curvature at the north pole by using the method from earlier. A small circle of radius ϵ centred at the north pole has a radius that is a distance obtained by increasing the polar angle θ from 0 at the pole to ϵ :

$$\text{radius} = \int_0^\epsilon d\theta = \epsilon \quad (2)$$

The circumference of this circle is

$$\text{circum} = \int_0^{2\pi} \sin \theta d\phi \quad (3)$$

$$= \sin \epsilon \int_0^{2\pi} d\phi \quad (4)$$

$$= 2\pi \sin \epsilon \quad (5)$$

The curvature is obtained from the formula

$$R = \lim_{\text{radius} \rightarrow 0} \frac{6}{\text{radius}^2} \left(1 - \frac{\text{circumference}}{2\pi \text{radius}} \right) \quad (6)$$

which in this case is

$$R = \lim_{\epsilon \rightarrow 0} \frac{6}{\epsilon^2} \left(1 - \frac{2\pi \sin \epsilon}{2\pi \epsilon} \right) \quad (7)$$

$$= \lim_{\epsilon \rightarrow 0} \frac{6}{\epsilon^2} \left(1 - \frac{\sin \epsilon}{\epsilon} \right) \quad (8)$$

By expanding the sine in a series we have

$$\frac{\sin \epsilon}{\epsilon} = \frac{1}{\epsilon} \left(\epsilon - \frac{\epsilon^3}{3!} + \dots \right) \quad (9)$$

$$= 1 - \frac{\epsilon^2}{6} + \dots \quad (10)$$

Inserting this into 8 we have

$$R = +1 \quad (11)$$

Since all points on a sphere are equivalent, this is the curvature of a unit sphere at any point.