DERIVATIVES OF A ONE-FORM FIELD IN POLAR COORDINATES

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Post date: 17 Dec 2022.

Consider the one-form field with components in rectangular coordinates:

$$\tilde{p}=\left[x^{2}+3y,y^{2}+3x\right] \tag{1}$$

We've looked at converting the one-form to polar coordinates, but we now wish to look at its derivatives in polar coordinates.

One way of doing this is to calculate the derivatives in rectangular coordinates and then use the transformation matrices to convert the result to polar. Thus we have

$$\begin{aligned}p_{x,x}&=2x\\p_{x,y}&=3\\p_{y,x}&=3\\p_{y,y}&=2y\end{aligned} \tag{2}$$

or, in matrix notation

$$\left[p_{\alpha,\beta}\right]=\left[\begin{array}{cc}2x&3\\3&2y\end{array}\right] \tag{3}$$

$$=\left[\begin{array}{cc}2r\cos\theta&3\\3&2r\sin\theta\end{array}\right] \tag{4}$$

The derivatives have two lower indexes, so the transformation is

$$p_{\alpha';\beta'}=\Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'}p_{\mu,\nu} \tag{5}$$

where the primed indices refer to polar and the unprimed to rectangular. The matrix is

$$\left[\Lambda^{\nu}{}_{\beta'}\right]=\left[\begin{array}{cc}\cos\theta&-r\sin\theta\\\sin\theta&r\cos\theta\end{array}\right] \tag{6}$$

Doing the matrix multiplications with Maple, we need to be careful of the order of the matrices. Note that we sum over the row indexes μ and ν of both Λ matrices, so the matrix product is

$$\left[p_{\alpha';\beta'}\right] = \left[\Lambda^{\mu}{}_{\alpha'}\right]^{T}\left[p_{\mu,\nu}\right]\left[\Lambda^{\nu}{}_{\beta'}\right] \tag{7}$$

where the T superscript denotes the transpose.

Doing this gives, after simplifying (we use the shorthand notation $c \equiv \cos\theta$ and $s \equiv \sin\theta$):

$$p_{r;r} = 2r\left(c^3+s^3\right)+6sc \tag{8}$$
$$p_{r;\theta} = 2r^2\left(s^2c-sc^2\right)+3r\left(c^2-s^2\right) \tag{9}$$
$$p_{\theta;r} = 2r^2\left(s^2c-sc^2\right)+3r\left(c^2-s^2\right) \tag{10}$$
$$p_{\theta;\theta} = 2r^3\left(s^2c+sc^2\right)-6r^2sc \tag{11}$$

We can also find these derivatives by using the Christoffel symbols for polar coordinates

$$\begin{aligned}\Gamma^{r}{}_{\mu\nu} &= \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \\ \Gamma^{\theta}{}_{\mu\nu} &= \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix}\end{aligned} \tag{12}$$

where μ,ν can take on the values r,θ.

In terms of Christoffel symbols, the derivatives are

$$\begin{aligned}p_{\alpha;\beta} &= \frac{\partial p_\alpha}{\partial x^\beta}-p_\gamma\Gamma^{\gamma}{}_{\alpha\beta} \\ &= p_{\alpha,\beta}-p_\gamma\Gamma^{\gamma}{}_{\alpha\beta}\end{aligned} \tag{13}$$

Note that the derivative on the LHS uses the semicolon (covariant derivative), while the derivative on the RHS uses the comma (ordinary derivative with respect to x^β). Also note that, in contrast to the covariant derivative of a vector field, the second term on the RHS has a minus sign instead of a plus sign.

Everything is to be expressed in polar coordinates. In particular, we need the polar form of $\tilde{p}$:

$$\tilde{p} = \left[r^2\left(c^3+s^3\right)+6rsc, r^3\left(cs^2-c^2s\right)+3r^2\left(c^2-s^2\right)\right]$$

The derivatives $p_{\alpha,\beta}$ are (from Maple)

$$p_{r,r} = 2r\left(c^3+s^3\right)+6sc \tag{14}$$
$$p_{r,\theta} = 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right) \tag{15}$$
$$p_{\theta,r} = 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right) \tag{16}$$
$$p_{\theta,\theta} = r^3\left(2sc^2+2s^2c-s^3-c^3\right)-12r^2sc \tag{17}$$

We can now calculate $p_{\alpha;\beta}$ using 13.

$$p_{r;r} = p_{r,r} - p_\gamma \Gamma^\gamma{}_{rr} \tag{18}$$

From 12, $\Gamma^r{}_{rr} = \Gamma^\theta{}_{rr} = 0$ so

$$p_{r;r} = p_{r,r} = 2r\left(c^3+s^3\right)+6sc \tag{19}$$

Comparing with 8, we see this agrees with the earlier result.

For $p_{r;\theta}$ we have

$$p_{r;\theta} = 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right)-p_\gamma\Gamma^\gamma{}_{r\theta} \tag{20}$$

From 12, $\Gamma^r{}_{r\theta} = 0$ and $\Gamma^\theta{}_{r\theta} = r^{-1}$,so we have

$$p_{r;\theta} = 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right)-\frac{1}{r}\left(r^3\left(cs^2-c^2s\right)+3r^2\left(c^2-s^2\right)\right) \tag{21}$$
$$= 2r^2\left(s^2c-sc^2\right)+3r\left(c^2-s^2\right) \tag{22}$$

This agrees with 9.

For $p_{\theta;r}$ we have

$$p_{\theta;r} = 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right)-p_\gamma\Gamma^\gamma{}_{\theta r} \tag{23}$$
$$= 3r^2\left(s^2c-sc^2\right)+6r\left(c^2-s^2\right)-\frac{1}{r}\left(r^3\left(cs^2-c^2s\right)+3r^2\left(c^2-s^2\right)\right) \tag{24}$$
$$= 2r^2\left(s^2c-sc^2\right)+3r\left(c^2-s^2\right) \tag{25}$$

which agrees with 10.

For $p_{\theta;\theta}$ we have

$$p_{\theta;\theta} = r^3\left(2sc^2+2s^2c-s^3-c^3\right)-12r^2sc-p_\gamma\Gamma^\gamma{}_{\theta\theta} \tag{26}$$
$$= r^3\left(2sc^2+2s^2c-s^3-c^3\right)-12r^2sc+r\left(r^2\left(c^3+s^3\right)+6rsc\right) \tag{27}$$
$$= 2r^3\left(s^2c+sc^2\right)-6r^2sc \tag{28}$$

which agrees with 11.

Finally, we note that the divergence of a one-form is undefined, since we cannot contract the derivatives of a one-form, due to the fact that both its indexes are lower indexes.