DERIVATIVES OF A VECTOR FIELD IN POLAR COORDINATES

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Consider the vector field with components in rectangular coordinates:

$$\vec{V}=\left[x^2+3y,y^2+3x\right] \tag{1}$$

We've looked at converting the vector to polar coordinates, but we now wish to look at its derivatives in polar coordinates.

One way of doing this is to calculate the derivatives in rectangular coordinates and then use the transformation matrices to convert the result to polar. Thus we have

$$\begin{aligned} V^x{}_{,x} &= 2x \\ V^x{}_{,y} &= 3 \\ V^y{}_{,x} &= 3 \\ V^y{}_{,y} &= 2y \end{aligned} \tag{2}$$

or,, in matrix notation

$$V^\alpha{}_{,\beta}=\left[\begin{array}{cc} 2x & 3 \\ 3 & 2y \end{array}\right] \tag{3}$$

$$=\left[\begin{array}{cc} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{array}\right] \tag{4}$$

The derivatives have one upper and one lower index, so we need both forms of the transformation matrix to do the conversion. That is

$$V^{\alpha'}{}_{;\beta}=\Lambda^{\alpha'}{}_{\gamma}\Lambda^{\delta}{}_{\beta'}V^{\gamma}{}_{,\delta} \tag{5}$$

where the primed indices refer to polar and the unprimed to rectangular. Note the use of a semicolon in $V^{\alpha'}{}_{;\beta}$. This indicates the *total* (covariant) derivative, which takes into account that the basis vectors in polar coordinates are not constant. In general, $V^{\alpha'}{}_{;\beta}\neq V^{\alpha'}{}_{,\beta}$. The two derivatives are equal only in coordinate systems where the basis vectors are constants, such as rectangular coordinates.

The matrices are

$$\Lambda^{\alpha'}{}_{\gamma} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \tag{6}$$

$$\Lambda^{\delta}{}_{\beta'} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{7}$$

Doing the matrix multiplications with Maple, we need to be careful of the order of the matrices. Note that we sum over the column index γ of $\Lambda^{\alpha'}{}_{\gamma}$ and the row index δ of $\Lambda^{\delta}{}_{\beta'}$, so we need to multiply the matrices in the order

$$\left[V^{\alpha'}{}_{;;\beta}\right] = \left[\Lambda^{\alpha'}{}_{\gamma}\right]\left[V^{\gamma}{}_{,\delta}\right]\left[\Lambda^{\delta}{}_{\beta'}\right] \tag{8}$$

Doing this gives the results

$$V^{r}{}_{;r} = \left(-2\cos(\theta)^2 r + 6\cos(\theta) + 2r\right)\sin(\theta) + 2\cos(\theta)^3 r \tag{9}$$

$$V^{r}{}_{;\theta} = -2r\left(\frac{3}{2} + \cos(\theta)^3 r + (r\sin(\theta) - 3)\cos(\theta)^2 - r\cos(\theta)\right) \tag{10}$$

$$V^{\theta}{}_{;r} = \frac{-3 - 2\cos(\theta)^3 r + (-2r\sin(\theta) + 6)\cos(\theta)^2 + 2r\cos(\theta)}{r} \tag{11}$$

$$V^{\theta}{}_{;\theta} = 2\cos(\theta)\sin(\theta)\left(r\cos(\theta) + r\sin(\theta) - 3\right) \tag{12}$$

We can also find these derivatives by using the Christoffel symbols for polar coordinates

$$\begin{aligned} \Gamma^{r}{}_{\mu\nu} &= \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \\ \Gamma^{\theta}{}_{\mu\nu} &= \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix} \end{aligned} \tag{13}$$

where μ,ν can take on the values r,θ.

In terms of Christoffel symbols, the derivatives are

$$V^{\alpha}{}_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\gamma}\Gamma^{\alpha}{}_{\gamma\beta} \tag{14}$$

where everything is to be expressed in polar coordinates. In particular, we need the polar form of $\vec{V}$

$$\begin{aligned} \vec{V}_{\text{polar}} = \big[& r^2\left(\cos^3\theta + \sin^3\theta\right) + 6r\sin\theta\cos\theta, \\ & r\left(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\right) + 3\left(\cos^2\theta - \sin^2\theta\right)\big] \end{aligned} \tag{15}$$

The derivatives $\frac{\partial V^{\alpha}}{\partial x^{\beta}}$ are (from Maple)

$$\frac{\partial V^r}{\partial r} = 2r\left(\cos^3\theta + \sin^3\theta\right) + 6\sin\theta\cos\theta \tag{16}$$

$$\frac{\partial V^r}{\partial \theta} = 3r^2\left(-\sin\theta\cos^2\theta + \sin^2\theta\cos\theta\right) + 6r\left(\cos^2\theta - \sin^2\theta\right) \tag{17}$$

$$\frac{\partial V^\theta}{\partial r} = \cos\theta\sin^2\theta - \cos^2\theta\sin\theta \tag{18}$$

$$\frac{\partial V^\theta}{\partial \theta} = r\left(2\sin\theta\cos^2\theta - \sin^3\theta - \cos^3\theta + 2\sin^2\theta\cos\theta\right) - 12\cos\theta\sin\theta \tag{19}$$

We can now calculate ${V^\alpha}_{,\beta}$ using 14.

$${V^r}_{,r} = \frac{\partial V^r}{\partial r} + V^\gamma {\Gamma^r}_{\gamma r} \tag{20}$$

From 13, ${\Gamma^r}_{rr} = {\Gamma^r}_{\theta r} = 0$ so

$${V^r}_{;r} = \frac{\partial V^r}{\partial r} = 2r\left(\cos^3\theta + \sin^3\theta\right) + 6\sin\theta\cos\theta \tag{21}$$

Comparing with 9, we can simplify the earlier expression using $\cos^2\theta + \sin^2\theta = 1$. In particular, the term

$$-2\cos(\theta)^2 r + 6\cos(\theta) + 2r = 2r\left(1 - \cos^2\theta\right) + 6\cos\theta \tag{22}$$

$$= 2r\sin^2\theta + 6\cos\theta \tag{23}$$

Inserting this back into 9 we see that it is the same as 21.

For ${V^r}_{,\theta}$ we have

$${V^r}_{;\theta} = 3r^2\left(-\sin\theta\cos^2\theta + \sin^2\theta\cos\theta\right) + 6r\left(\cos^2\theta - \sin^2\theta\right) + V^\gamma {\Gamma^r}_{\gamma\theta} \tag{24}$$

From 13, ${\Gamma^r}_{r\theta} = 0$ and ${\Gamma^r}_{\theta\theta} = -r$,so we have

$$\begin{aligned} {V^r}_{;\theta} &= 3r^2\left(-\sin\theta\cos^2\theta + \sin^2\theta\cos\theta\right) + 6r\left(\cos^2\theta - \sin^2\theta\right) + \\ &\quad - r\left(r\left(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\right) + 3\left(\cos^2\theta - \sin^2\theta\right)\right) \qquad (25) \\ &= -2r\left(\frac{3}{2} + \cos(\theta)^3 r + \left(r\sin(\theta) - 3\right)\cos(\theta)^2 - r\cos(\theta)\right) \qquad (26) \end{aligned}$$

where we used Maple to simplify the expression and get the last line. This agrees with 10.

For ${V^\theta}_r$ we have

$$V^{\theta}{}_{;r} = \cos\theta\sin^2\theta - \cos^2\theta\sin\theta + V^{\theta}\Gamma^{\theta}{}_{\theta r} \tag{27}$$
$$= \cos\theta\sin^2\theta - \cos^2\theta\sin\theta +$$
$$\frac{1}{r}\left[r\left(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\right) + 3\left(\cos^2\theta - \sin^2\theta\right)\right] \tag{28}$$
$$= \frac{-3 - 2\cos(\theta)^3 r + (-2r\sin(\theta) + 6)\cos(\theta)^2 + 2r\cos(\theta)}{r} \tag{29}$$

which agrees with 11.

Finally, for $V^{\theta}{}_{,\theta}$ we have

$$V^{\theta}{}_{;\theta} = r\left(2\sin\theta\cos^2\theta - \sin^3\theta - \cos^3\theta + 2\sin^2\theta\cos\theta\right) - 12\cos\theta\sin\theta + V^r\Gamma^{\theta}{}_{r\theta} \tag{30}$$
$$= r\left(2\sin\theta\cos^2\theta - \sin^3\theta - \cos^3\theta + 2\sin^2\theta\cos\theta\right) - 12\cos\theta\sin\theta +$$
$$\frac{1}{r}\left[r^2\left(\cos^3\theta + \sin^3\theta\right) + 6r\sin\theta\cos\theta\right] \tag{31}$$
$$= 2\cos(\theta)\sin(\theta)\left(r\cos(\theta) + r\sin(\theta) - 3\right) \tag{32}$$

which agrees with 12.

Most of these expressions can be converted into other (possibly nicer) forms using trig identities, but the purpose of this exercise is to show that we can get the same expressions for the derivatives by converting the rectangular to the polar form using the Λ matrices, and by using the Christoffel symbols on the polar forms directly.

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