DIAGONALIZING THE METRIC

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Post date: 12 Mar 2023.

Most of the metrics encountered in introductions to general relativity are diagonal, in the sense that the spacetime element is given as

$$ds^2 = g_{\alpha\beta}(x)\,dx^\alpha dx^\beta \tag{1}$$

and the matrix $\left[g_{\alpha\beta}\right]$ is diagonal. The local flatness theorem tells us that, in any manifold, it is possible to approximate the metric at any point $\mathcal{P}$ in spacetime by $\eta_{\alpha\beta}$ to first order in the distance. That is

$$g_{\alpha\beta}(x^\mu) = \eta_{\alpha\beta} + \mathcal{O}\left((x^\mu - \mathcal{P}^\mu)^2\right) \tag{2}$$

This implies that any metric can be approximated by a diagonal metric at any given point. An important point is that the approximation is valid only in the neighbourhood of a particular point $\mathcal{P}$, and if we move away from this point, we may have to find a different approximation. This is analogous to the situation where we approximate a curve at a given point by its tangent line; the tangent varies as we move along the curve, unless the curve is actually a straight line.

One way of demonstrating this is by diagonalizing a metric $g'_{\alpha\beta}(x)$. Since the metric is symmetric, a theorem from matrix algebra states that it is always possible to find an orthonormal matrix P such that

$$P^T g' P = g \tag{3}$$

where $g = \left[g_{\alpha\beta}\right]$ is a diagonal matrix. The term *orthonormal* means that the columns of P are mutually orthogonal unit vectors. This theorem applies here since, at any given point in spacetime, the metric's components are just numbers, rather than functions of position, so the diagonalization procedure can be applied directly.

To apply the method, we will, in general, also need to transform the coordinates by a linear transformation. That is, we will need to apply

$$x^\alpha = M^\alpha{}_\beta x'^\beta \tag{4}$$

for some matrix M.

The key to finding P and M is to observe that the infinitesimal interval must have the same value in any coordinate system, so we must have

$$ds^2 = ds'^2 \tag{5}$$

or

$$g'_{\alpha\beta}\left(x'\right)dx'^{\alpha}dx'^{\beta} = g_{\alpha\beta}\left(x\right)dx^{\alpha}dx^{\beta} \tag{6}$$

We can now insert the transformations 3 and 4 to get

$$P^{\gamma}{}_{\alpha}g'_{\gamma\epsilon}P^{\epsilon}{}_{\beta}M^{\alpha}{}_{\theta}dx'^{\theta}M^{\beta}{}_{\zeta}dx'^{\zeta} = g'_{\alpha\beta}dx'^{\alpha}dx'^{\beta} \tag{7}$$

We can regroup the terms on the LHS to get

$$\left(P^{\gamma}{}_{\alpha}M^{\alpha}{}_{\theta}\right)\left(P^{\epsilon}{}_{\beta}M^{\beta}{}_{\zeta}\right)g'_{\gamma\epsilon}dx'^{\theta}dx'^{\zeta} = g'_{\alpha\beta}dx'^{\alpha}dx'^{\beta} \tag{8}$$

In order for the two sides to be equal for arbitrary displacements, we must therefore have

$$\begin{aligned} P^{\gamma}{}_{\alpha}M^{\alpha}{}_{\theta} &= \delta^{\gamma}{}_{\theta} \\ P^{\epsilon}{}_{\beta}M^{\beta}{}_{\zeta} &= \delta^{\epsilon}{}_{\zeta} \end{aligned} \tag{9}$$

That is, in terms of the matrices P and M, we must have

$$M = P^{-1} \tag{10}$$

Thus if we can find the matrix P that diagonalizes g', we can then find the required linear transformation M of the coordinates.

There is a standard procedure (to be found in linear algebra textbooks) for diagonalizing a real symmetric matrix, so we can follow that to get P, and then take the inverse to get M. We'll give an example to see how this works.

Example 1. Suppose we're given the metric

$$ds'^2 = -dt'^2 + 2dt'\,dx' + dy'^2 + dz'^2 \tag{11}$$

and we wish to diagonalize it. Since the y and z coordinates are already diagonal, we can focus on t and x, and use a 2×2 matrix.

$$g' = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

The procedure for diagonalizing a symmetric matrix involves finding the eigenvalues and eigenvectors. I've used Maple for this, but you can do it by hand if you like. The eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \tag{13}$$

with eigenvectors

$$\vec{v} = \begin{bmatrix} \frac{2}{1+\sqrt{5}/2} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{1-\sqrt{5}/2} \\ 1 \end{bmatrix} \tag{14}$$

These are not unit vectors, so we need to normalize them (by dividing each vector by its length), and we get for P:

$$P = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} & -\frac{2}{\sqrt{10-2\sqrt{5}}} \\ \frac{\sqrt{2}\left(\sqrt{5}+1\right)}{2\sqrt{5+\sqrt{5}}} & \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{bmatrix} \tag{15}$$

[You can check at this point that the columns are also orthogonal, although it's a bit messy so you can take my word for it - I checked it in Maple.]

If we now calculate 3 we get

$$g = P^T g' P = \begin{bmatrix} \frac{10+10\sqrt{5}}{\left(5+\sqrt{5}\right)^2} & 0 \\ 0 & \frac{10-10\sqrt{5}}{\left(\sqrt{5}-5\right)^2} \end{bmatrix} \tag{16}$$

which is indeed diagonal. Note also that one term is positive and the other negative, which preserves the signature of the metric (positive for space and negative for time).

The linear transformation 4 can be found by taking the inverse of P (again, using Maple as it's a bit of a mess):

$$M = P^{-1} = \begin{bmatrix} \frac{\left(\sqrt{5}-1\right)\sqrt{2}\sqrt{5}\sqrt{5+\sqrt{5}}}{20} & \frac{\sqrt{5}\sqrt{2}\sqrt{5+\sqrt{5}}}{10} \\ -\frac{\left(\sqrt{5}+1\right)\sqrt{5}\sqrt{10-2\sqrt{5}}}{20} & \frac{\sqrt{5}\sqrt{10-2\sqrt{5}}}{10} \end{bmatrix} \tag{17}$$

Thus the new set of coordinates is given by

$$x = Mx' = \begin{bmatrix} \frac{\left(\sqrt{5}-1\right)\sqrt{2}\sqrt{5}\sqrt{5+\sqrt{5}}t'}{20} + \frac{\sqrt{5}\sqrt{2}\sqrt{5+\sqrt{5}}x'}{10} \\ -\frac{\left(\sqrt{5}+1\right)\sqrt{5}\sqrt{10-2\sqrt{5}}t'}{20} + \frac{\sqrt{5}\sqrt{10-2\sqrt{5}}x'}{10} \end{bmatrix} \tag{18}$$

Using 16 and 18 we can verify that

$$g_{\alpha\beta} dx^\alpha \, dx^\beta = -dt'^2 + 2dx' \, dt' \tag{19}$$

At this point, it's worth noting that there is actually a much simpler solution to this particular example. If we start with 11 and propose a transformation to a diagonal metric, we can write the new coordinates x and t as

$$\begin{aligned} t &= Ax' + Bt' \\ x &= Cx' + Dt' \end{aligned} \tag{20}$$

Then we have

$$dt^2 = A^2 dx'^2 + 2AB dx'\, dt' + B^2 dt'^2 \tag{21}$$

$$dx^2 = C^2 dx'^2 + 2CD dx'\, dt' + D^2 dt'^2 \tag{22}$$

To require that the new metric (over x and t) is diagonal, we require that

$$ds^2 = -dt^2 + dx^2 \tag{23}$$

$$= \left(C^2 - A^2\right) dx'^2 + 2\left(CD - AB\right) dx'\, dt' + \left(D^2 - B^2\right) dt'^2 \tag{24}$$

This must equal 19, so by comparing terms, we have

$$C^2 - A^2 = 0 \tag{25}$$

$$CD - AB = 1 \tag{26}$$

$$D^2 - B^2 = -1 \tag{27}$$

One solution of these equations is

$$\begin{aligned} A &= -1 \\ B &= C = 1 \\ D &= 0 \end{aligned} \tag{28}$$

so the required transformation is

$$t = t' - x' \tag{29}$$

$$x = x' \tag{30}$$

This gives the transformation matrix as

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{31}$$

and the diagonalizing matrix as

$$P = M^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{32}$$

We can verify that

$$g = P^T g' P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

Thus the diagonalizing matrix is not unique.