DIVERGENCE IN CURVED SPACE

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In Chapter I.5, Appendix 3 of *Einstein Gravity*, Zee derives a formula giving the divergence of a vector field in curved space. The formula is

$$D_\mu W^\mu \equiv \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}W^\mu\right) \tag{1}$$

$$= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\right)W^\mu + \partial_\mu W^\mu \tag{2}$$

where g is the determinant of the metric tensor.

For spherical coordinates, we have

$$g_{\mu\nu} = \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta \end{array}\right] \tag{3}$$

so

$$\sqrt{g} = r^2\sin\theta \tag{4}$$

If we apply 2 to spherical coordinates, we get

$$D_\mu W^\mu = \frac{1}{r^2\sin\theta}\left[2r\sin\theta W^r + r^2\cos\theta W^\theta\right] + \partial_r W^r + \partial_\theta W^\theta + \partial_\phi W^\phi \tag{5}$$

$$= \frac{2}{r}W^r + \frac{\cos\theta}{\sin\theta}W^\theta + \partial_r W^r + \partial_\theta W^\theta + \partial_\phi W^\phi \tag{6}$$

Although this result agrees with the formula quoted in Zee's Appendix 3, it does not agree with the usual formula for the divergence in spherical coordinates (see, for example, the end papers in Griffiths *Introduction to Electrodynamics*), which is

$$\nabla\cdot W = \frac{1}{r^2}\frac{\partial\left(r^2W^r\right)}{\partial r} + \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta W^\theta\right)}{\partial\theta} + \frac{\partial W^\phi}{\partial\phi}\right] \tag{7}$$

The difference between the two formulas is due to the choice of basis vectors for spherical coordinates. In Zee's formula 6, the basis vectors are in the usual orthogonal directions of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, but these vectors have lengths given by the metric tensor, which are $\sqrt{g_{ii}}$, so we have

$$|\hat{\boldsymbol{r}}| = 1 \tag{8}$$

$$\left|\hat{\boldsymbol{\theta}}\right| = r \tag{9}$$

$$\left|\hat{\boldsymbol{\phi}}\right| = r\sin\theta \tag{10}$$

The usual form for the divergence has basis vectors in the same directions, but they are unit vectors. As a result, we need to scale each component of the vector field by the length of its corresponding basis vector in Zee's equation. That is, we need to calculate

$$D_\mu W^\mu = \sum_\mu \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\frac{W^\mu}{\sqrt{g_{\mu\mu}}}\right) \tag{11}$$

where I've inserted an explicit summation sign to indicate that the index μ is summed only once. Making this change, we have

$$D_\mu W^\mu = \frac{1}{r^2\sin\theta}\left[2r\sin\theta W^r + r^2\cos\theta\frac{W^\theta}{r}\right] + \partial_r W^r + \frac{1}{r}\partial_\theta W^\theta + \frac{1}{r\sin\theta}\partial_\phi W^\phi \tag{12}$$

$$= \frac{1}{r^2}\frac{\partial\left(r^2 W^r\right)}{\partial r} + \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta W^\theta\right)}{\partial\theta} + \frac{\partial W^\phi}{\partial\phi}\right] \tag{13}$$

Thus the result now agrees with the usual formula 7.

Note that in Zee's formula 6, if we had used the unit vectors as the basis, then the units in the formula are inconsistent, in that the terms in W^r have units of inverse length (times whatever the units of the vector field W^μ are), while the other terms are dimensionless. When we divide by $\sqrt{g_{\mu\mu}}$, we introduce an inverse length into the two angular components, making the overall units consistent.