DIVERGENCE OF VECTOR FIELD IN POLAR COORDINATES

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We consider the vector field with components in rectangular coordinates:

$$\vec{V} = \left[x^2 + 3y, y^2 + 3x\right] \tag{1}$$

The divergence of a vector field $V^{\alpha}{}_{,\alpha}$ is a scalar, so should be the same in all coordinate systems. In rectangular coordinates, we have

$$\begin{aligned} V^{x}{}_{,x} &= 2x \\ V^{x}{}_{,y} &= 3 \\ V^{y}{}_{,x} &= 3 \\ V^{y}{}_{,y} &= 2y \end{aligned} \tag{2}$$

or,, in matrix notation

$$V^{\alpha}{}_{,\beta} = \left[\begin{array}{cc} 2x & 3 \\ 3 & 2y \end{array}\right] \tag{3}$$

$$= \left[\begin{array}{cc} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{array}\right] \tag{4}$$

The divergence is

$$V^{\alpha}{}_{,\alpha} = 2(x+y) = 2r(\cos\theta + \sin\theta) \tag{5}$$

To calculate it in polar coordinates, we use our earlier results. Note that for polars we use the semicolon to indicate a covariant derivative since the basis vectors are not constants.

$$V^{r}{}_{;r} = 2r\left(\cos^3\theta + \sin^3\theta\right) + 6\sin\theta\cos\theta \tag{6}$$

$$= 2r\left(\cos^2\theta\cos\theta + \sin^2\theta\sin\theta\right) + 6\sin\theta\cos\theta \tag{7}$$

$$V^{\theta}{}_{;\theta} = 2\cos(\theta)\sin(\theta)\left(r\cos(\theta) + r\sin(\theta) - 3\right) \tag{8}$$

$$= 2r\left(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta\right) - 6\sin\theta\cos\theta \tag{9}$$

The sum is

$$V^{\beta'}{}_{;\beta'} = 2r\left(\cos^2\theta\cos\theta + \sin^2\theta\sin\theta + \cos^2\theta\sin\theta + \cos\theta\sin^2\theta\right) \tag{10}$$
$$= 2r\left[\left(\cos^2\theta + \sin^2\theta\right)\left(\cos\theta + \sin\theta\right)\right] \tag{11}$$
$$= 2r\left(\cos\theta + \sin\theta\right) \tag{12}$$
$$= 2\left(x + y\right) \tag{13}$$

Thus the divergence is indeed the same in both rectangular and polar coordinates.

We can also use the standard formula for calculating divergence in polars:

$$\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}\left(rV^r\right) + \frac{\partial}{\partial\theta}V^\theta \tag{14}$$

To use this, we need the components of $\vec{V}$ in polar cooordinates

$$\vec{V} = \left[\begin{array}{c} r^2\left(\cos(\theta)^3 + \sin(\theta)^3\right) + 6r\sin(\theta)\cos(\theta) \\ r\left(\cos(\theta)\sin(\theta)^2 - \cos(\theta)^2\sin(\theta)\right) + 3\cos(\theta)^2 - 3\sin(\theta)^2 \end{array}\right] \tag{15}$$

The derivatives get quite messy, so I used Maple to calculate and simplify them. We have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rV^r\right) = r\left(\cos(\theta)^3 + \sin(\theta)^3\right) + 6\sin(\theta)\cos(\theta) + \\ 2r\left(\cos(\theta)^3 + \sin(\theta)^3\right) + 6\cos(\theta)\sin(\theta) \tag{16}$$

$$\frac{\partial}{\partial\theta}V^\theta = r\left(-\sin(\theta)^3 + 2\cos(\theta)^2\sin(\theta) + 2\cos(\theta)\sin(\theta)^2 - \cos(\theta)^3\right) - \\ 12\cos(\theta)\sin(\theta) \tag{17}$$

Simplifying does indeed give

$$\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}\left(rV^r\right) + \frac{\partial}{\partial\theta}V^\theta = 2r\left(\cos\theta + \sin\theta\right) \tag{18}$$