EINSTEIN EQUATION - ALTERNATIVE FORM

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Post date: 25 Jan 2023.

The general relativistic generalization of Newton's law of gravity is

$$G^{ij}+\Lambda g^{ij}=\kappa T^{ij} \tag{1}$$

where the Einstein tensor is defined in terms of the Ricci tensor and the curvature scalar as

$$G^{ij}\equiv R^{ij}-\frac{1}{2}g^{ij}R \tag{2}$$

We can write this in a different form that is sometimes easier to use in calculations. Eliminating G^{ij} we have

$$R^{ij}-\frac{1}{2}g^{ij}R+\Lambda g^{ij}=\kappa T^{ij} \tag{3}$$

Multiplying both sides by g_{ij} we get

$$g_{ij}R^{ij}-\frac{1}{2}g_{ij}g^{ij}R+\Lambda g_{ij}g^{ij}=\kappa g_{ij}T^{ij} \tag{4}$$

Because the tensor g_{ij} is the inverse of g^{ij}, their product gives the identity matrix of rank 4 (this can be seen by doing the calculation in a local inertial frame where $g_{ij}=\eta_{ij}$ and noting that since it's a tensor equation, it's valid in all coordinate systems). That is

$$g_{ij}g^{jk}=\delta_i^k \tag{5}$$

so if we contract the δ_j^k tensor we just sum up its diagonal elements and since these are all 1 (and there are four rows), we get

$$\delta_k^k=4 \tag{6}$$

Returning to 4 we get

$$g_{ij}R^{ij}-2R+4\Lambda=\kappa g_{ij}T^{ij} \tag{7}$$

Since the curvature scalar is given by

$$R\equiv g_{ij}R^{ij} \tag{8}$$

and the stress-energy scalar is

$$T \equiv g_{ij} T^{ij} \tag{9}$$

we get

$$-R + 4\Lambda = \kappa T \tag{10}$$

Multiplying this by $-\frac{1}{2} g^{ij}$ and subtracting from 3 we have

$$R^{ij} - \Lambda g^{ij} = \kappa \left(T^{ij} - \frac{1}{2} g^{ij} T \right) \tag{11}$$

Isolating the Ricci tensor gives us the alternative form of the Einstein equation:

$$\boxed{R^{ij} = \kappa \left(T^{ij} - \frac{1}{2} g^{ij} T \right) + \Lambda g^{ij}} \tag{12}$$

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