EINSTEIN EQUATION IN THE NEWTONIAN LIMIT

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The Einstein equation is

$$R^{ij}=\kappa\left(T^{ij}-\frac{1}{2}g^{ij}T\right)+\Lambda g^{ij} \tag{1}$$

where we have yet to determine the constant κ. To do this, we need to show that the Einstein equation reduces to Newton's law of gravity for weak gravitational fields. Actually, there are three conditions that should hold in the Newtonian limit. First, as we've said, the gravitational field is weak, meaning that spacetime is nearly flat. Second, objects should travel with a speed much less than the speed of light (the spatial four-velocity components $u^i\ll 1$ for $i=x,y,z$). The second condition implies that the only non-negligible component of the stress-energy tensor T^{ij} is T^{tt}. For example, for a perfect fluid, we can assume that it's effectively at rest, so the off-diagonal elements are all zero. For the diagonal spatial components, we have (for T^{zz}; the other 2 components have the analogous formulas):

$$T^{zz}=\frac{1}{L^3}\int dp^x\int dp^y\int\left(p^z\right)^2\frac{N(p)}{p^t}dp^z \tag{2}$$

This equation is for a cubic volume of side length L containing $N(p)$ particles of momentum p. Since $p^z=mu^z$ and $u^z\ll 1$, $T^{zz}\approx 0$ (the requirement of a weak gravitational field means that $N(p)$ can't be very large, since we can't have that much mass). As the spatial diagonal elements $T^{ii}=P$, the pressure and $T^{tt}=\rho$, the energy density, this condition translates to $\rho\gg P$.

With these assumptions, we can try to show that the relativistic equation of geodesic deviation reduces to the Newtonian version. That is, we want to show that

$$\ddot{\mathbf{n}}^i=-R^i_{\;j\ell m}u^m u^j n^\ell \tag{3}$$

reduces to

$$\ddot{n}^i=-\eta^{ij}\left(\partial_k\partial_j\Phi\right)n^k\text{ (for }i,j,k=x,y,z) \tag{4}$$

where n^i is the separation of two infinitesimally close geodesics (this is the tidal force.) and Φ is the Newtonian gravitational potential.

Starting with 3, we can eliminate terms where $m \neq t$ or $j \neq t$ (because $u^i \approx 0$ for $i \neq t$) to get

$$\ddot{n}^i = -R^i_{\ t\ell t} n^\ell \tag{5}$$

where we're now considering only the spatial components: $i, \ell = x, y, z$. Note that summing ℓ over x, y, z is the same as summing it over t, x, y, z since due to the anti-symmetry of the Riemann tensor under interchange of its last two indices, $R^i_{\ ttt} = -R^i_{\ ttt} = 0$. Also, because we're in the non-relativistic limit, the proper time and coordinate time are essentially the same thing: $\tau \approx t$, so the time derivative is with respect to t.

Comparing this to 4, we have (renaming ℓ to k in the last equation):

$$R^i_{\ tkt} \approx \eta^{ij} \left(\partial_k \partial_j \Phi\right) \tag{6}$$

Newton's law of gravity in differential form is

$$\begin{aligned} \nabla^2 \Phi &= 4\pi G \rho & (7)\\ &= \eta^{ij} \partial_i \partial_j \Phi & (8)\\ &\approx R^i_{\ tit} & (9)\\ &= R_{tt} & (10) \end{aligned}$$

(Again, the contraction over index i in the second to last line can be taken over 3 or 4 coordinates, since $R^i_{\ ttt} = 0$.) We can raise both indices on the Ricci tensor in the usual way:

$$\begin{aligned} R^{tt} &= g^{ti} g^{tj} R_{ij} & (11)\\ &\approx \eta^{ti} \eta^{tj} R_{ij} & (12)\\ &= (-1)^2 R_{tt} & (13)\\ &= R_{tt} & (14) \end{aligned}$$

so we can write 10 in upper index form as

$$\nabla^2 \Phi \approx R^{tt} \tag{15}$$

Now looking back at 1 and using the condition that $T^{tt} = \rho$ is the only significant entry in the stress-energy tensor, we have

$$T = g_{ij} T^{ij} = \eta_{ij} T^{ij} = -\rho \tag{16}$$

so

$$R^{tt} = \kappa\left(T^{tt} - \frac{1}{2}\eta^{tt}T\right) + \Lambda\eta^{tt} \tag{17}$$

$$= \frac{\kappa}{2}\rho - \Lambda \tag{18}$$

$$\approx \nabla^2\Phi \tag{19}$$

$$= 4\pi G\rho \tag{20}$$

Comparing the second and fourth lines, we see that $\Lambda \approx 0$ and

$$\kappa = 8\pi G \tag{21}$$

and the Einstein equation becomes

$$R^{ij} = 8\pi G\left(T^{ij} - \frac{1}{2}g^{ij}T\right) \tag{22}$$

or in its original form

$$G^{ij} = 8\pi G T^{ij} \tag{23}$$

Actually, we can't take $\Lambda = 0$; all we can say is that for gravitational systems on the scale of the solar system (where Newtonian theory works well, except in the case of the orbit of Mercury) $\Lambda \ll 4\pi G\rho$. To get a feel for how small Λ needs to be, suppose we have a spherical gravitational potential in empty space ($\rho = 0$). Then in spherical coordinates

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) \tag{24}$$

$$\approx -\Lambda \tag{25}$$

$$\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) \approx -\Lambda r^2 \tag{26}$$

$$\frac{d\Phi}{dr} \approx -\frac{\Lambda}{3}r \tag{27}$$

This is the radial component (the only non-zero component) of the gradient of the potential, and the negative gradient of the gravitational potential is the gravitational field, which is the acceleration of gravity. In the solar system, Newton's theory says that the acceleration due to the sun is

$$g = \frac{GM_{\odot}}{r^2} \tag{28}$$

so if $\Lambda \neq 0$ but its effect is not felt in Newton's theory, we must have

$$\frac{\Lambda}{3}r \ll \frac{GM_{\odot}}{r^2} \tag{29}$$

in order for Newton's theory to be valid in the solar system. Distances in the solar system are around $r \approx 10^{12}$ m, $GM_{\odot} \approx 1500$ m and G in general relativistic units is 7.426×10^{-28} m kg^{-1} so this means

$$\frac{\Lambda}{8\pi G} \ll \frac{3 \times 1500}{8\pi\left(7.426 \times 10^{-28}\right)\left(10^{12}\right)^3} = 2.4 \times 10^{-7}\ \text{kg m}^{-3} \tag{30}$$

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