EINSTEIN EQUATION ON THE SURFACE OF A SPHERE

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According to the Einstein equation, the Riemann tensor in 2D must be zero in empty space, implying that gravitational fields cannot exist in 2D. Another consequence of the Einstein equation is that the stress-energy must be zero on the surface of a sphere. That is, even though a 2D surface is manifestly curved, the curvature is not the result of any mass or energy. This is another example of how general relativity breaks down in two dimensions.

The Einstein equation is

$$R^{ij} = \kappa\left(T^{ij} - \frac{1}{2}g^{ij}T\right) + \Lambda g^{ij} \tag{1}$$

The Ricci tensor for a spherical surface is

$$R^{ij} = \left[\begin{array}{cc} \frac{1}{r^4} & 0 \\ 0 & \frac{1}{r^4\sin^2\theta} \end{array}\right] \tag{2}$$

The metric for a sphere is (in both forms):

$$g^{ij} = \left[\begin{array}{cc} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\theta} \end{array}\right] \tag{3}$$

$$g_{ij} = \left[\begin{array}{cc} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{array}\right] \tag{4}$$

Since the off-diagonal elements of g^{ij} and R^{ij} are all zero, 1 tells us that

$$T^{\theta\phi} = T^{\phi\theta} = 0 \tag{5}$$

To deal with the diagonal elements, we first need the stress-energy scalar.

$$T = g_{ij}T^{ij} \tag{6}$$

$$= r^2 T^{\theta\theta} + r^2\sin^2\theta T^{\phi\phi} \tag{7}$$

We have

$$
\begin{aligned}
R^{\theta\theta} &= \frac{1}{r^4} && (8)\\
&= \kappa\left(T^{\theta\theta} - \frac{1}{2r^2}T\right) + \frac{\Lambda}{r^2} && (9)\\
R^{\phi\phi} &= \frac{1}{r^4\sin^2\theta} && (10)\\
&= \kappa\left(T^{\phi\phi} - \frac{1}{2r^2\sin^2\theta}T\right) + \frac{\Lambda}{r^2\sin^2\theta} && (11)
\end{aligned}
$$

Combining these we get

$$
\begin{aligned}
\frac{R^{\theta\theta}}{\sin^2\theta} - R^{\phi\phi} = \kappa\left(\frac{T^{\theta\theta}}{\sin^2\theta} - T^{\phi\phi}\right) = 0 && (12)\\
\frac{T^{\theta\theta}}{\sin^2\theta} - T^{\phi\phi} = 0 && (13)\\
T^{\theta\theta} = T^{\phi\phi}\sin^2\theta && (14)\\
T = 2r^2\sin^2\theta T^{\phi\phi} && (15)
\end{aligned}
$$

Therefore

$$
T^{\theta\theta} - \frac{1}{2r^2}T = T^{\phi\phi} - \frac{1}{2r^2\sin^2\theta}T = 0 \tag{16}
$$

holds identically. Thus the stress-energy contribution to the Einstein equation is always zero on a sphere (although the stress-energy *tensor* may have two non-zero components, these two components always combine to give zero contribution to the Einstein equation).