EINSTEIN EQUATION SOLUTION FOR THE INTERIOR OF A SPHERICALLY SYMMETRIC STAR

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The derivation of the Schwarzschild metric applies to the empty space outside a spherically symmetric source. We can apply a similar method to try to find a metric for the interior of a spherically symmetric, static source such as a non-rotating star. As usual, we can make a number of simplifying assumptions. Spherically symmetry implies that the metric has the general form

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

where the functions A and B are to be determined; they depend only on r because we're assuming that everything is independent of time. Another simplification is the assumption that the star's matter is a perfect fluid. We've looked at the Einstein equation for a perfect fluid before, which has the general form (with $\Lambda = 0$):

$$R^{\mu\nu} = 8\pi G\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right) \tag{2}$$

The stress-energy tensor is

$$T^{\mu\nu} = (\rho + P)\,u^\mu u^\nu + P g^{\mu\nu} \tag{3}$$

where u^μ is the four-velocity, ρ is the energy density and P is the pressure, both of which can be functions of r.

We have for the stress-energy scalar:

$$T = (\rho + P)\,g_{\mu\nu}u^\mu u^\nu + P g_{\mu\nu}g^{\mu\nu} \tag{4}$$

$$= -(\rho + P) + 4P \tag{5}$$

$$= 3P - \rho \tag{6}$$

since $g_{\mu\nu}g^{\mu\nu} = 4$ and $g_{\mu\nu}u^\mu u^\nu = -1$ in any coordinate system. This gives

$$T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = (\rho + P)\,u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}(\rho - P) \tag{7}$$

Lowering the indices in 7, we get

$$T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T=(\rho+P)\,u_{\mu}u_{\nu}+\frac{1}{2}g_{\mu\nu}(\rho-P) \tag{8}$$

Since the fluid is at rest, the spatial components of the four-velocity u_i are all zero. From the condition $g_{\mu\nu}u^{\mu}u^{\nu}=-1$ we have

$$g_{\mu\nu}u^{\mu}u^{\nu}=g_{tt}u^{t}u^{t}=-A\left(u^{t}\right)^{2}=-1 \tag{9}$$

$$u^{t}=\frac{1}{\sqrt{A}} \tag{10}$$

$$u_{t}=g_{t\nu}u^{\nu}=g_{tt}u^{t}=-Au^{t}=-\sqrt{A} \tag{11}$$

Equation 2 with indices lowered is

$$R_{\mu\nu}=8\pi G\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) \tag{12}$$

so with the above results, we get

$$R_{tt}=8\pi G\left((\rho+P)\,u_{t}u_{t}+\frac{1}{2}g_{tt}(\rho-P)\right) \tag{13}$$

$$=8\pi G\left[(\rho+P)\,A-\frac{A}{2}(\rho-P)\right] \tag{14}$$

$$=4\pi GA(\rho+3P) \tag{15}$$

$$R_{rr}=8\pi G\left((\rho+P)\,u_{r}u_{r}+\frac{1}{2}g_{rr}(\rho-P)\right) \tag{16}$$

$$=4\pi G(0+B(\rho-P)) \tag{17}$$

$$=4\pi GB(\rho-P) \tag{18}$$

$$R_{\theta\theta}=8\pi G\left((\rho+P)\,u_{\theta}u_{\theta}+\frac{1}{2}g_{\theta\theta}(\rho-P)\right) \tag{19}$$

$$=4\pi G\left(0+r^{2}(\rho-P)\right) \tag{20}$$

$$=4\pi Gr^{2}(\rho-P) \tag{21}$$

We can combine these three equations to eliminate P:

$$\frac{R_{tt}}{A}+\frac{R_{rr}}{B}+\frac{2R_{\theta\theta}}{r^{2}}=16\pi G\rho \tag{22}$$

When we worked out the Ricci tensor in terms of the metric, we got the equations

$$\frac{1}{2B}\left[\partial_{rr}^{2}A-\partial_{tt}^{2}B+\frac{(\partial_t B)^2}{2B}+\frac{(\partial_t A)(\partial_t B)-(\partial_r A)^2}{2A}-\frac{(\partial_r A)(\partial_r B)}{2B}+\frac{2\partial_r A}{r}\right]=R_{tt} \tag{23}$$

$$\frac{1}{2A}\left[\partial_{tt}^{2}B-\partial_{rr}^{2}A+\frac{(\partial_r A)^2-(\partial_t A)(\partial_t B)}{2A}+\frac{(\partial_r A)(\partial_r B)-(\partial_t B)^2}{2B}+\frac{2A\partial_r B}{rB}\right]=R_{rr} \tag{24}$$

$$-\frac{r\partial_r A}{2AB}+\frac{r\partial_r B}{2B^2}+1-\frac{1}{B}=R_{\theta\theta} \tag{25}$$

$$\frac{\partial_t B}{rB}=R_{tr} \tag{26}$$

All time derivatives are zero, so these equations simplify to

$$\frac{1}{2B}\left[\partial_{rr}^{2}A-\frac{(\partial_r A)^2}{2A}-\frac{(\partial_r A)(\partial_r B)}{2B}+\frac{2\partial_r A}{r}\right]=R_{tt} \tag{27}$$

$$\frac{1}{2A}\left[-\partial_{rr}^{2}A+\frac{(\partial_r A)^2}{2A}+\frac{(\partial_r A)(\partial_r B)}{2B}+\frac{2A\partial_r B}{rB}\right]=R_{rr} \tag{28}$$

$$-\frac{r\partial_r A}{2AB}+\frac{r\partial_r B}{2B^2}+1-\frac{1}{B}=R_{\theta\theta} \tag{29}$$

$$0=R_{tr} \tag{30}$$

Applying 22 we get

$$\frac{\partial_r A}{ABr}+\frac{\partial_r B}{rB^2}-\frac{\partial_r A}{rAB}+\frac{\partial_r B}{rB^2}+\frac{2}{r^2}-\frac{2}{r^2 B}=16\pi G\rho(r) \tag{31}$$

$$r\frac{\partial_r B}{B^2}+1-\frac{1}{B}=8\pi G r^2\rho(r) \tag{32}$$

$$\frac{d}{dr}\left[r\left(1-\frac{1}{B}\right)\right]=8\pi G r^2\rho(r) \tag{33}$$

$$r\left(1-\frac{1}{B}\right)=2G\int_0^r 4\pi\left(r'\right)^2\rho\left(r'\right)dr' \tag{34}$$

The integral on the last line resembles the mass of the star out to radius r, as it's the integral of the density ρ over a set of spherical shells of radius r', surface area $4\pi\left(r'\right)^2$ and thickness dr'. However, the Schwarzschild radial coordinate r isn't equal to the actual radial distance (it's a circumferential coordinate), so this isn't quite accurate. With the definition

$$m(r) \equiv \int_0^r 4\pi \left(r'\right)^2 \rho\left(r'\right) dr' \tag{35}$$

we therefore have

$$r\left(1-\frac{1}{B}\right) = 2Gm(r) \tag{36}$$

$$B = \left[1-\frac{2Gm(r)}{r}\right]^{-1} \tag{37}$$

This gives us B, but what about A? If we use the stress-energy tensor's conservation equation

$$\nabla_\nu T^{\mu\nu} = 0 \tag{38}$$

we can look at $\mu = r$ component, starting from 3. The absolute gradient is defined in terms of Christoffel symbols as

$$\nabla_\rho T^{\mu\nu} = \partial_\rho T^{\mu\nu} + T^{\mu\alpha}\Gamma^\nu_{\alpha\rho} + T^{\alpha\nu}\Gamma^\mu_{\alpha\rho} \tag{39}$$

so for $\mu = r$ and $\rho = \nu$ we have

$$\nabla_\nu T^{r\nu} = \partial_\nu T^{r\nu} + T^{\alpha\nu}\Gamma^r_{\alpha\nu} + T^{r\alpha}\Gamma^\nu_{\alpha\nu} \tag{40}$$

Since the covariant derivative of the metric tensor is zero and P is a scalar (so its covariant derivative is the same as its ordinary derivative), we have

$$\nabla_\nu T^{r\nu} = \nabla_\nu\left[(\rho+P)u^r u^\nu\right] + \nabla_\nu\left(Pg^{r\nu}\right) \tag{41}$$

$$= \nabla_\nu\left[(\rho+P)u^r u^\nu\right] + g^{rr}\partial_r P \tag{42}$$

$$= \partial_\nu\left[(\rho+P)u^r u^\nu\right] + (\rho+P)\left[u^\alpha u^\nu \Gamma^r_{\alpha\nu} + u^r u^\alpha \Gamma^\nu_{\alpha\nu}\right] + g^{rr}\partial_r P \tag{43}$$

$$= 0 \tag{44}$$

Because the fluid is at rest, $u^r = u^\theta = u^\phi = 0$ so this reduces to

$$(\rho+P)u^t u^t \Gamma^r_{tt} + g^{rr}\partial_r P = 0 \tag{45}$$

For a diagonal metric $g^{\mu\nu} = \frac{1}{g_{\mu\nu}}$ so $g^{rr} = \frac{1}{B}$, from 10 $u^t = \frac{1}{\sqrt{A}}$, and from the Christoffel symbol worksheet $\Gamma^r_{tt} = \Gamma^1_{00} = \frac{1}{2B}\partial_r A$ so we have

$$(\rho+P)\frac{\partial_r A}{2AB}+\frac{\partial_r P}{B}=0 \tag{46}$$

$$\frac{1}{A}\frac{dA}{dr}=-\frac{2}{\rho+P}\frac{dP}{dr} \tag{47}$$

To solve this, we need to know both ρ and P as functions of r. We can make some headway by using $R_{\theta\theta}$ in 21 and 25. We have

$$-\frac{r\partial_r A}{2AB}+\frac{r\partial_r B}{2B^2}+1-\frac{1}{B}=4\pi Gr^2(\rho-P) \tag{48}$$

From 32 and 36

$$r\frac{\partial_r B}{B^2}=8\pi Gr^2\rho-\left(1-\frac{1}{B}\right) \tag{49}$$

$$=8\pi Gr^2\rho-\frac{2Gm}{r} \tag{50}$$

Plugging this and 37 into 48:

$$-\frac{r}{2B}\left(-\frac{2}{\rho+P}\frac{dP}{dr}\right)+4\pi Gr^2\rho-\frac{Gm}{r}+\frac{2Gm}{r}=4\pi Gr^2(\rho-P) \tag{51}$$

$$\frac{r^2}{\rho+P}\left(1-\frac{2Gm}{r}\right)\frac{dP}{dr}=-\left(4\pi Gr^3P+Gm\right) \tag{52}$$

$$\frac{dP}{dr}=-\frac{(\rho+P)}{r^2}\frac{\left(4\pi Gr^3P+Gm\right)}{1-2Gm/r} \tag{53}$$

This is the Oppenheimer-Volkoff equation for stellar structure. In order to solve it, we need a couple of other equations involving m and ρ, but we'll leave that for later. [By the way, for anyone interested in trivia, the Volkoff after whom this equation is named is George Volkoff, and he was the Dean of Science at the University of British Columbia in Vancouver when I was doing my undergraduate degree in physics and astronomy there in the 1970s. Sadly, I never had him as a professor.]