ELECTROMAGNETIC FIELD TENSOR - LORENTZ TRANSFORMATIONS

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Post date: 25 Aug 2022.

The electromagnetic field tensor is

$$F^{ij}=\left[\begin{array}{cccc} 0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{1}$$

We can use the usual tensor transformation rules to see how the electric and magnetic fields transform under a Lorentz transformation. We get

$$F'^{ij}=\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}F^{kl} \tag{2}$$

$$=\Lambda^i{}_k\Lambda^j{}_lF^{kl} \tag{3}$$

where the Lorentz transformation matrix is, for a boost in the x direction with velocity β:

$$\Lambda^i{}_k=\left[\begin{array}{cccc} \gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

As we saw when discussing the inertia tensor, we can write this transformation as a matrix equation

$$F'=\Lambda F\Lambda^T \tag{5}$$

The first product is

$$\Lambda F=\left[\begin{array}{cccc} \gamma\beta E_x & \gamma E_x & \gamma E_y-\gamma\beta B_z & \gamma E_z+\gamma\beta B_y\\ -\gamma E_x & -\gamma\beta E_x & -\gamma\beta E_y+\gamma B_z & -\gamma\beta E_z-\gamma B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{6}$$

The final product is

$$F'=\Lambda F\Lambda^{T}=\left[\begin{array}{cccc} 0 & \gamma^{2}\left(1-\beta^{2}\right)E_{x} & \gamma E_{y}-\gamma\beta B_{z} & \gamma E_{z}+\gamma\beta B_{y}\\ -\gamma^{2}\left(1-\beta^{2}\right)E_{x} & 0 & -\gamma\beta E_{y}+\gamma B_{z} & -\gamma\beta E_{z}-\gamma B_{y}\\ -\gamma E_{y}+\gamma\beta B_{z} & \gamma\beta E_{y}-\gamma B_{z} & 0 & B_{x}\\ -\gamma E_{z}-\gamma\beta B_{y} & \gamma\beta E_{z}+\gamma B_{y} & -B_{x} & 0 \end{array}\right] \tag{7}$$

Using $\gamma=1/\sqrt{1-\beta^{2}}$ we get

$$F'=\left[\begin{array}{cccc} 0 & E_{x} & \gamma E_{y}-\gamma\beta B_{z} & \gamma E_{z}+\gamma\beta B_{y}\\ -E_{x} & 0 & -\gamma\beta E_{y}+\gamma B_{z} & -\gamma\beta E_{z}-\gamma B_{y}\\ -\gamma E_{y}+\gamma\beta B_{z} & \gamma\beta E_{y}-\gamma B_{z} & 0 & B_{x}\\ -\gamma E_{z}-\gamma\beta B_{y} & \gamma\beta E_{z}+\gamma B_{y} & -B_{x} & 0 \end{array}\right] \tag{8}$$

From this, we see that

$$E'_{x}=E_{x} \tag{9}$$
$$E'_{y}=\gamma E_{y}-\gamma\beta B_{z} \tag{10}$$
$$E'_{z}=\gamma E_{z}+\gamma\beta B_{y} \tag{11}$$
$$B'_{x}=B_{x} \tag{12}$$
$$B'_{y}=\gamma\beta E_{z}+\gamma B_{y} \tag{13}$$
$$B'_{z}=-\gamma\beta E_{y}+\gamma B_{z} \tag{14}$$

Unlike lengths, the components of E and B in the direction of motion are unchanged, while those perpendicular to the motion are altered.

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