ELECTROMAGNETIC FIELD TENSOR - CHANGE IN KINETIC ENERGY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 24 Aug 2022.

In the last post, we introduced the electromagnetic field tensor F^{ij}:

$$F^{ij}=\left[\begin{array}{cccc} 0 & E_{x} & E_{y} & E_{z}\\ -E_{x} & 0 & B_{z} & -B_{y}\\ -E_{y} & -B_{z} & 0 & B_{x}\\ -E_{z} & B_{y} & -B_{x} & 0 \end{array}\right] \tag{1}$$

In terms of F^{ij}, the electric and magnetic (Lorentz) force laws for a charge q can be combined into a single equation:

$$\frac{dp^{i}}{d\tau}=qF^{ij}\eta_{ja}u^{a} \tag{2}$$

where $u^{a}=\gamma\left[1,v_{x},v_{y},v_{z}\right]$ is the four-velocity. The three spatial components give the force law, but what about the time component? The time component of the four-momentum is the relativistic energy which, for a particle of rest mass m is γm. If we expand this in a Taylor series for small v, we get

$$\gamma m=m+\frac{1}{2}mv^{2}+\ldots \tag{3}$$

Thus the relativistic energy is the rest mass plus the Newtonian kinetic energy (plus higher order terms). In the small-v limit, $\gamma\to1$ and the t component of 2 therefore is

$$\frac{d}{d\tau}\left(m+\frac{1}{2}mv^{2}\right) = q\left[\gamma E_{x}v_{x}+\gamma E_{y}v_{y}+\gamma E_{z}v_{z}\right] \tag{4}$$

$$= q\gamma\mathbf{v}\cdot\mathbf{E} \tag{5}$$

$$\approx q\mathbf{v}\cdot\mathbf{E} \tag{6}$$

Since the rest mass doesn't change, this equation is saying that the rate of change of kinetic energy is $q\mathbf{v}\cdot\mathbf{E}$. Does this make sense?

The force on a charge in an electric field is $q\mathbf{E}$, so the work done by this field in moving the charge through a distance $d\mathbf{r}$ is $q\left(d\mathbf{r}\cdot\mathbf{E}\right)$. This work accelerates the charge, thus increasing its kinetic energy. The rate at which the kinetic energy increases is therefore $q\frac{d}{dt}(d\mathbf{r}\cdot\mathbf{E})=q\mathbf{v}\cdot\mathbf{E}$.

Since the magnetic force on a charge is always perpendicular to the direction of motion, magnetic forces do no work, so there is no contribution to the kinetic energy from the magnetic field.