

ELECTROMAGNETIC FIELD TENSOR - CONSERVATION OF MASS

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The electric and magnetic fields \mathbf{E} and \mathbf{B} can be combined into a single tensor known as the *electromagnetic field tensor* F^{ij} :

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (1)$$

We can see from its definition that this tensor is anti-symmetric, that is, that $F^{ij} = -F^{ji}$. For any anti-symmetric tensor we can show that

$$F^{ij}\eta_{ia}\eta_{jb}u^au^b = 0 \quad (2)$$

In this equation, η_{ij} is the metric tensor in flat space and u^a is the four-velocity, but in fact the formula is valid for any tensors η and u , provided that F is anti-symmetric. The proof involves a bit of index-switching.

$$F^{ij}\eta_{ia}\eta_{jb}u^au^b = -F^{ji}\eta_{ia}\eta_{jb}u^au^b \quad (3)$$

$$= -F^{ij}\eta_{ja}\eta_{ib}u^au^b \quad (4)$$

$$= -F^{ij}\eta_{jb}\eta_{ia}u^bu^a \quad (5)$$

In the second line, we swapped the dummy indexes i and j , and in the third line we swapped a and b . The result shows that the original quantity is equal to its negative, which means it must be zero.

In terms of F^{ij} , the electric and magnetic (Lorentz) force laws for a charge q can be combined into a single equation:

$$\frac{dp^i}{d\tau} = qF^{ij}\eta_{ja}u^a \quad (6)$$

where $u^a = \gamma[1, v_x, v_y, v_z]$ is the four-velocity.

For example, if $i = 1$ we get

$$\frac{dp^1}{d\tau} = q\gamma(E_x + v_y B_z - v_z B_y) \quad (7)$$

In the non-relativistic limit, $\gamma \rightarrow 1$ and this is the x component of the force law $\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$. We'll explore some of the other properties of this tensor later.

Since the square of the four-momentum of a particle is the negative of its mass squared ($\mathbf{p} \cdot \mathbf{p} = \gamma^2 m^2 (-1 + v^2) = -m^2$), this should be conserved for a charged particle moving in an electromagnetic field. (Its *total* momentum is not conserved since the fields exert a force on the particle.)

We have

$$\frac{d(\mathbf{p} \cdot \mathbf{p})}{d\tau} = \frac{d}{d\tau} (\eta_{ij} p^i p^j) \quad (8)$$

$$= \eta_{ij} \left[\frac{dp^i}{d\tau} p^j + p^i \frac{dp^j}{d\tau} \right] \quad (9)$$

$$= 2\eta_{ij} \frac{dp^i}{d\tau} p^j \quad (10)$$

$$= 2q\eta_{ij} F^{ik} \eta_{ka} u^a p^j \quad (11)$$

$$= 2qm F^{ik} \eta_{ij} \eta_{ka} u^a u^j \quad (12)$$

$$= 0 \quad (13)$$

In the third line, we used the fact that $\eta_{ij} = \eta_{ji}$ and swapped i and j in the second term. The fourth line uses 6 and the last line uses 2.

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