

## ELECTROMAGNETIC FIELD TENSOR - CONSERVATION OF MASS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 23 Aug 2022.

The electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  can be combined into a single tensor known as the *electromagnetic field tensor*  $F^{ij}$ :

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (1)$$

We can see from its definition that this tensor is anti-symmetric, that is, that  $F^{ij} = -F^{ji}$ . For any anti-symmetric tensor we can show that

$$F^{ij}\eta_{ia}\eta_{jb}u^au^b = 0 \quad (2)$$

In this equation,  $\eta_{ij}$  is the metric tensor in flat space and  $u^a$  is the four-velocity, but in fact the formula is valid for any tensors  $\eta$  and  $u$ , provided that  $F$  is anti-symmetric. The proof involves a bit of index-switching.

$$F^{ij}\eta_{ia}\eta_{jb}u^au^b = -F^{ji}\eta_{ia}\eta_{jb}u^au^b \quad (3)$$

$$= -F^{ij}\eta_{ja}\eta_{ib}u^au^b \quad (4)$$

$$= -F^{ij}\eta_{jb}\eta_{ia}u^bu^a \quad (5)$$

In the second line, we swapped the dummy indexes  $i$  and  $j$ , and in the third line we swapped  $a$  and  $b$ . The result shows that the original quantity is equal to its negative, which means it must be zero.

In terms of  $F^{ij}$ , the electric and magnetic (Lorentz) force laws for a charge  $q$  can be combined into a single equation:

$$\frac{dp^i}{d\tau} = qF^{ij}\eta_{ja}u^a \quad (6)$$

where  $u^a = \gamma[1, v_x, v_y, v_z]$  is the four-velocity.

For example, if  $i = 1$  we get

$$\frac{dp^1}{d\tau} = q\gamma(E_x + v_y B_z - v_z B_y) \quad (7)$$

In the non-relativistic limit,  $\gamma \rightarrow 1$  and this is the  $x$  component of the force law  $\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$ . We'll explore some of the other properties of this tensor later.

Since the square of the four-momentum of a particle is the negative of its mass squared ( $\mathbf{p} \cdot \mathbf{p} = \gamma^2 m^2 (-1 + v^2) = -m^2$ ), this should be conserved for a charged particle moving in an electromagnetic field. (Its *total* momentum is not conserved since the fields exert a force on the particle.)

We have

$$\frac{d(\mathbf{p} \cdot \mathbf{p})}{d\tau} = \frac{d}{d\tau} (\eta_{ij} p^i p^j) \quad (8)$$

$$= \eta_{ij} \left[ \frac{dp^i}{d\tau} p^j + p^i \frac{dp^j}{d\tau} \right] \quad (9)$$

$$= 2\eta_{ij} \frac{dp^i}{d\tau} p^j \quad (10)$$

$$= 2q\eta_{ij} F^{ik} \eta_{ka} u^a p^j \quad (11)$$

$$= 2qm F^{ik} \eta_{ij} \eta_{ka} u^a u^j \quad (12)$$

$$= 0 \quad (13)$$

In the third line, we used the fact that  $\eta_{ij} = \eta_{ji}$  and swapped  $i$  and  $j$  in the second term. The fourth line uses 6 and the last line uses 2.

#### PINGBACKS

Pingback: [Electromagnetic field tensor - change in kinetic energy](#)

Pingback: [Electromagnetic field tensor - contractions with metric tensor](#)

Pingback: [Electromagnetic field tensor - invariance under Lorentz transformations](#)

Pingback: [Electromagnetic field tensor - a couple of Maxwell's equations](#)

Pingback: [Electromagnetic field tensor - Lorentz transformations](#)

Pingback: [Electromagnetic field tensor - justification](#)

Pingback: [Electromagnetic field tensor - four-potential](#)

Pingback: [Electromagnetic Lorentz invariant](#)

Pingback: [Maxwell's equations using the electromagnetic field tensor](#)

Pingback: [Maxwell's equations from the electromagnetic field tensor](#)