ELECTROMAGNETIC STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 7; Problem 7.6.

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Since energy is the time component of the four-momentum and the density of a scalar quantity such as mass or charge is the time component of a four-vector (four-current in the case of charge), the energy density of a collection of particles would appear to transform as the tt component of a second-rank tensor (since we have one Lorentz transformation for the energy and another for the density). If we then take a slight leap in logic and assume that the energy density of electromagnetic fields transforms in the same way, then we can assume that this energy density is the tt component of a tensor known as the *stress-energy tensor*.

To deduce the form of this tensor, we use the usual technique of trying to convert an expression from classical physics into tensor form. In this case the classical expression for energy density is

$$\rho_E = \frac{1}{8\pi k}\left(E^2 + B^2\right) \tag{1}$$

The idea, then, is to find a second-rank tensor from the electromagnetic field tensor that has this as its tt component. We can get a clue as to how to do this by looking at the quantity $F_{ij}F^{ij} = 2\left(B^2 - E^2\right)$. During the derivation of this, we found that the tt (that is, the 00 component) component of the tensor product is $F^{0j}F_{j0} = E^2$. Combining this with $B^2 = \frac{1}{2}F_{ij}F^{ij} + E^2 = \frac{1}{2}F_{ij}F^{ij} + F^{0j}F_{j0}$ we get

$$E^2 + B^2 = 2F^{0j}F_{j0} + \frac{1}{2}F_{ij}F^{ij} \tag{2}$$

If we raise the 0 index in the first term on the RHS, we get

$$E^2 + B^2 = -2F^{0j}F_{jk}\eta^{k0} + \frac{1}{2}F_{ij}F^{ij} \tag{3}$$

where we've introduced the minus sign in the first term since raising the index changes the sign in the 0 column of F_{jk}.

This looks promising, but the expression on the RHS is the sum of the tt component of a tensor (the first term) and a scalar. We can convert this to be the tt component of a tensor by multiplying the last term by $-\eta^{00} = +1$:

$$E^2 + B^2 = -2F^{0j}F_{jk}\eta^{k0} - \frac{1}{2}F_{ij}F^{ij}\eta^{00} \tag{4}$$

If we now propose this is the tt term of the stress-energy tensor $8\pi k T^{ml}$ (where $k = \frac{1}{4\pi\epsilon_0}$ is the constant used by Moore) we can postulate

$$T^{ml} = -\frac{1}{8\pi k}\left(2F^{mj}F_{jk}\eta^{kl} + \frac{1}{2}F_{ij}F^{ij}\eta^{ml}\right) \tag{5}$$

$$= -\epsilon_0 F^{mj}F_{jk}\eta^{kl} - \frac{\epsilon_0}{4}F_{ij}F^{ij}\eta^{ml} \tag{6}$$

The other elements of T^{ml} have meanings that we'll get to eventually, but we can note that for $i \neq 0$

$$T^{0i} = -\epsilon_0 F^{0j}F_{jk}\eta^{ki} \tag{7}$$

For $i = 1$ for example, we get

$$T^{01} = -\epsilon_0 F^{0j}F_{j1} \tag{8}$$

$$= \epsilon_0 \left(E_y B_z - E_z B_y\right) \tag{9}$$

$$= \epsilon_0 \left(\mathbf{E} \times \mathbf{B}\right)_x \tag{10}$$

This is the x component of the Poynting vector, which describes the rate of energy transfer per unit area of an electromagnetic field.