EMBEDDING 2-D CURVED SPACE IN 3-D - THE SPHERE

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Post date: 9 Sep 2022.

We can now look at some examples of embedding a curved-space 2-d metric in 3-d flat space. We'll begin with the familiar case of the spherical surface. However, the goal here is to start with a metric defined in terms of some unknown coordinates and then discover the nature of the surface by embedding it in 3-d space.

The metric is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{1}$$

To get this in the form where we can embed it using cylindrical coordinates (cylindrical coordinates are actually 'flat', since we can cut the cylinder parallel to its axis and roll it out to get a plane), we need the ϕ term to be $r^2 d\phi^2$ for some coordinate r. Therefore, we can try defining $r = R\sin\theta$ and see what this does to the θ component. We get

$$dr = R\cos\theta d\theta \tag{2}$$

$$d\theta = \frac{dr}{R\cos\theta} \tag{3}$$

$$= \frac{dr}{R\sqrt{1-\sin^2\theta}} \tag{4}$$

$$= \frac{dr}{R\sqrt{1-\left(\frac{r}{R}\right)^2}} \tag{5}$$

Thus the metric can be rewritten in terms of r and ϕ as follows:

$$ds^2 = \frac{dr^2}{1-\left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \tag{6}$$

From here, we equate this to the 3-d cylindrical metric:

$$dz^2 + r^2 d\phi^2 + dr^2 = \frac{dr^2}{1 - \left(\frac{r}{R}\right)^2} + r^2 d\phi^2 \tag{7}$$

$$\frac{dz}{dr} = \pm \frac{r/R}{\sqrt{1 - \left(\frac{r}{R}\right)^2}} \tag{8}$$

$$z = \pm R \sqrt{1 - \left(\frac{r}{R}\right)^2} \tag{9}$$

$$= \pm \sqrt{R^2 - r^2} \tag{10}$$

This is the equation of the two halves (upper and lower) of a sphere, so we see that the embedding of the 2-d metric in 3-d space does indeed give a sphere.