EMBEDDING A 2-D CURVED SURFACE INTO 3-D - INVERSE COSH

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 9 Sep 2022.

A final example embedding a curved-space2-d metric in 3-d flat space. This time, our 2-d metric is

$$ds^2 = d\rho^2 + \left(\rho^2 + b^2\right) d\phi^2 \tag{1}$$

where b is a constant.

To convert the ϕ component to the required form of $r^2 d\phi^2$, we define $r = \sqrt{\rho^2 + b^2}$. Then

$$\begin{aligned} dr &= \frac{\rho}{\sqrt{\rho^2 + b^2}} d\rho && (2) \\ &= \frac{\sqrt{r^2 - b^2}\, d\rho}{r} && (3) \\ d\rho &= \frac{r}{\sqrt{r^2 - b^2}} dr && (4) \end{aligned}$$

The metric becomes:

$$ds^2 = \frac{r^2 dr^2}{r^2 - b^2} + r^2 d\phi^2 \tag{5}$$

We now equate this to the cylindrical metric:

$$\begin{aligned} dz^2 + r^2 d\phi^2 + dr^2 &= \frac{r^2 dr^2}{r^2 - b^2} + r^2 d\phi^2 && (6) \\ \frac{dz}{dr} &= \frac{b}{\sqrt{r^2 - b^2}} && (7) \end{aligned}$$

This integral can be done with software or looked up, and is

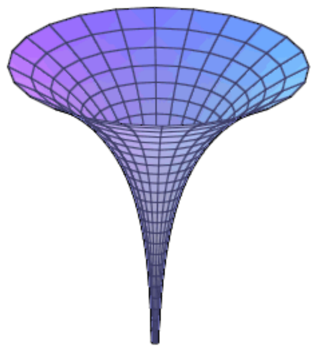

FIGURE 1. The surface $z = b\text{arcosh}\left(\frac{r}{b}\right)$.

$$z = b\ln\left(r + \sqrt{r^2 - b^2}\right) \tag{8}$$

$$= b\ln\left[b\left(\frac{r}{b} + \sqrt{\frac{r^2}{b^2} - 1}\right)\right] \tag{9}$$

$$= b\ln\left(\frac{r}{b} + \sqrt{\frac{r^2}{b^2} - 1}\right) + b\ln b \tag{10}$$

From tables of inverse hyperbolic functions, we see that this is equivalent to

$$z = b\text{arcosh}\left(\frac{r}{b}\right) + b\ln b \tag{11}$$

We can ignore the last term as it is just a constant and serves only to raise or lower the surface as a whole. The surface (Fig. 1) is similar to Flamm's paraboloid.