EMBEDDING A TWO-DIMENSIONAL SURFACE IN THREE-DIMENSIONAL SPACE

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Given a metric in two dimensions, we can visualize it by embedding it in three-dimensional space. The general procedure for doing this can be fairly complicated, but we'll consider a simplified case here. [This is the same example as the one done in the previous post, but slightly more general.]

Suppose we have a metric in the form

$$d\Sigma^2 = f(r)\,dr^2 + g(r)\,d\phi^2 \tag{1}$$

where f and g are some arbitrary functions of the radial coordinate r.

The key point in this metric is that it doesn't depend on ϕ, so it is axially symmetric. That is, it doesn't depend on the axial angle ϕ. This suggests that if we are to embed the surface Σ in 3-d space, we might try cylindrical coordinates, since a function with axial symmetry is represented as a surface of revolution about the z axis in the cylindrical coordinates $[\rho,\psi,z]$. Here ρ is the perpendicular distance from the z axis to a point on the surface, ψ is the axial angle and z is the vertical distance above the plane $z=0$.

The 3-d distance element in cylindrical coordinates is

$$dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \tag{2}$$

Given axial symmetry, $\psi=\phi$, $\rho=\rho(r)$ and $z=z(r)$, so taking differentials

$$\begin{aligned} d\rho &= \frac{d\rho}{dr}dr \\ dz &= \frac{dz}{dr}dr \\ d\psi &= d\phi \end{aligned} \tag{3}$$

so plugging these into 2 we have

$$dS^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\phi^2 \tag{4}$$

Comparing with 1 we have

$$\rho^2 = g(r) \tag{5}$$

$$\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = f(r) \tag{6}$$

We can use 5 to eliminate ρ in terms of r and substitute this into 6, which gives us an ordinary differential equation for z in terms of r. If this ODE can be solved (numerically if necessary), we can then plot z as a function of ρ to see the visualization of the 2-d surface in 3-d flat space.

Example 1. We're given the 2-d metric

$$d\Sigma^2 = \frac{dr^2}{1-2M/r} + r^2 d\phi^2 \tag{7}$$

To see what this surface looks like, we follow the above procedure. We have

$$\rho^2 = r^2 \tag{8}$$

$$\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1-2M/r} \tag{9}$$

We can take $\rho = r$ (since radial distances must be positive) which gives

$$\frac{d\rho}{dr} = 1 \tag{10}$$

and

$$1 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1-2M/r} \tag{11}$$

or

$$\frac{dz}{dr} = \sqrt{\frac{1}{1-2M/r} - 1} \tag{12}$$

$$= \sqrt{\frac{2M}{r-2M}} \tag{13}$$

We can integrate this to get

$$z = 2\sqrt{2M(r-2M)} \tag{14}$$

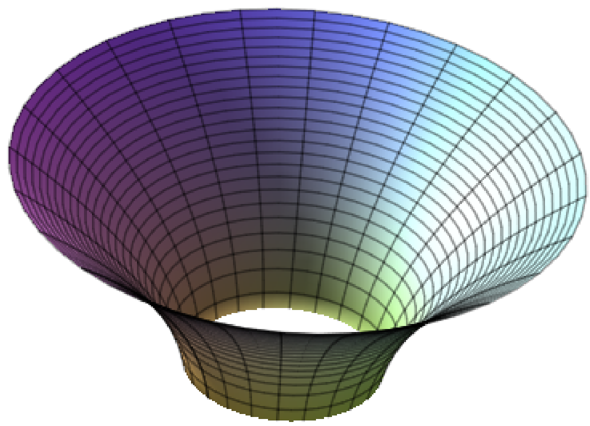

FIGURE 1. The surface $z=2\sqrt{2M(r-2M)}$ for $M=1$ and $r=[2,6]]$.

Since $\rho=r$, we can plot this function in 3-d cylindrical coordinates to get Fig. 1, a surface known as Flamm's paraboloid. It is actually a slice of the Schwarzchild geometry at constant time t and constant ϕ. The slope of the surface becomes infinite at $r=2M$.

PINGBACKS

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