EQUALITY OF GRAVITATIONAL AND INERTIAL MASSES

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Post date: 1 Mar 2023.

A central tenet of general relativity is the *principle of equivalence*. This states that an object in a uniform gravitational field is equivalent to the same object in free space undergoing a constant acceleration. One subtle but important implication of this principle is that an object's *gravitational* mass is the same as its *inertial* mass.

The distinction between the gravitational and inertial masses is often skimmed over in elementary physics courses but, if you think about it, it's a fundamental assumption even in Newtonian physics. To see why, consider an object such as a planet of mass m in a circular orbit of radius R about a star of mass M. The planet is held in its orbit by the gravitational attraction between the planet and star, which is given by Newton's formula

$$F_G = \frac{GmM}{R^2} \tag{1}$$

An object in uniform circular motion experiences a centripetal acceleration given by

$$a = \frac{v^2}{R} \tag{2}$$

where v is the object's constant speed in its orbit. The acceleration is due to a centripetal force which is, by Newton's law, the mass times the acceleration, so we have

$$F_I = \frac{mv^2}{R} \tag{3}$$

But wait a minute! The situation we're considering is that of a planet orbiting a star, so we can say that the centripetal force is due to the gravitational force. But we could equally well imagine a case where an object is held in a circular orbit by a different force, such as a stone being whirled around on a string. In this latter case, gravity doesn't play a part, yet the object's mass m still appears in the formula for F_I. In this case, the mass is the *inertial* mass, or the mass which resists the action of the force by means on its inertia. We have no apriori reason to believe that this inertial mass, which we'll call m_I, is the same as the mass m_G that appears in the gravitational force 1. The usual elementary physics calculation for determining the speed v in a circular orbit is

$$F_I = F_G \tag{4}$$

$$\frac{mv^2}{R} = \frac{GmM}{R^2} \tag{5}$$

which gives rise to

$$v = \sqrt{\frac{GM}{R}} \tag{6}$$

In step 5, we implicitly assumed that the planet's mass is the same on both sides. We should have written

$$\frac{m_I v^2}{R} = \frac{Gm_G M}{R^2} \tag{7}$$

so that

$$v = \sqrt{\frac{Gm_G M}{m_I R}} \tag{8}$$

The classic experiment designed to test if $m_I = m_G$ was done originally by Baron Eötvös, a Hungarian physicist (although earlier experiments were done by Bessel and by Newton himself, although they used different methods). Eötvös's experiment was designed to test if the ratio m_G/m_I was the same for objects of different compositions. It used a torsion balance, in which a dumbbell shape with the weights on either end composed of different substances was suspended by a thin fibre in such a way that the two weights balanced each other. If the experiment is done at some latitude on the Earth other than the equator, the weights on either end of the dumbbell will experience a gravitational force towards the centre of the Earth, and a centripetal force towards the axis of Earth's rotation, the latter due to the rotation of the Earth. A particularly sensitive experiment was done by Y. Su and collaborators in 1994 in Seattle, USA. This experiment was more sophisticated than the one described here, but the principle was the same. The weights were composed of copper and beryllium.

A simplified version of the experiment is shown in Fig. 1, where the centripetal force $\vec{F}_I$ is shown much larger than to scale in order to make it visible. The large arc represents the surface of the Earth in a vertical cross section. The gravitational force $\vec{F}_G$ is the green arrow pointing to the centre of the Earth. The centripetal force $\vec{F}_I$ is the blue arrow, which

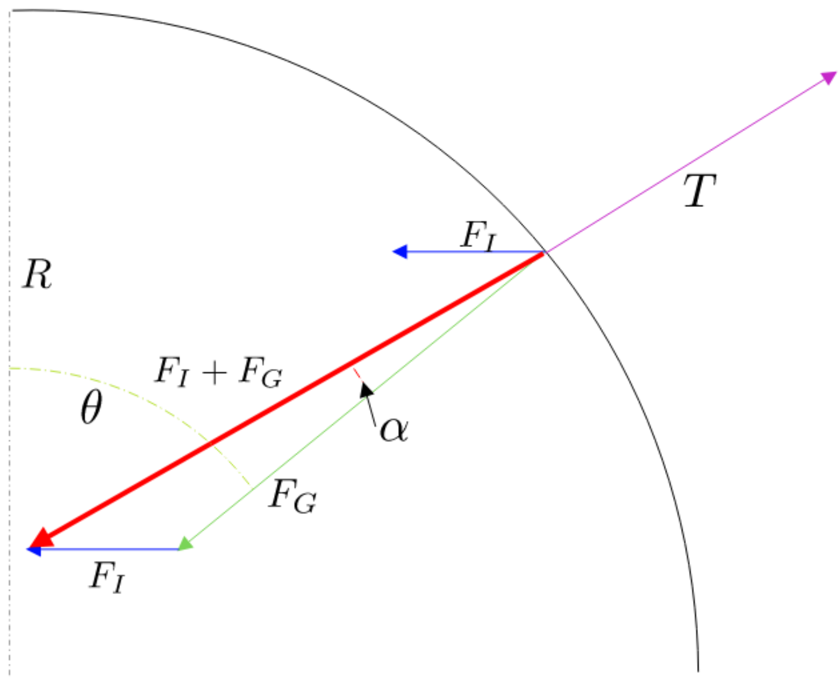

FIGURE 1. A simplified torsion balance experiment. Not to scale.

points horizontally in the same plane as that of the circle of latitude. The net force on the weight is the vector sum $\vec{F}_I + \vec{F}_G$, and is shown as the red arrow. The tension force $\vec{T}$ in the fibre (purple arrow) balances this net force. The magnitude of $\vec{T}$ equals the magnitude of $\vec{F}_I + \vec{F}_G$ (although it's shown shorter in the figure to conserve space). The angle θ is $\frac{\pi}{2} - \ell$, where ℓ is the latitude. R is the radius of the Earth, and P is its rotation period of 1 day, so we have

$$\begin{aligned} R &= 6.371 \times 10^6 \text{ m} \\ P &= 8.64 \times 10^4 \text{ s} \\ g &= 9.8 \text{ m s}^{-2} \end{aligned} \tag{9}$$

The orbital speed of the Earth at angle θ is then

$$v = \frac{2\pi R \sin\theta}{P} = 463.3 \sin\theta \text{ m s}^{-1} \tag{10}$$

We can work out the angle α between the gravitational force $\vec{F}_G$ and the tension $\vec{T}$ as follows. Introduce an xy coordinate system with x horizontal and y vertical as usual. Then

$$\vec{F}_I = -m_I \frac{v^2}{R}\hat{x} = -0.03369 m_I \sin\theta\hat{x} \quad (11)$$

$$\vec{F}_G = 9.8 m_G (-\sin\theta\hat{x} - \cos\theta\hat{y}) \quad (12)$$

The angle α that the fibre makes with the vertical is the angle between $\vec{F}_G$ and $\vec{F}_I + \vec{F}_G$, which we can get using the dot product:

$$\cos\alpha = \frac{\left(\vec{F}_I + \vec{F}_G\right) \cdot \vec{F}_G}{\left|\vec{F}_I + \vec{F}_G\right| \left|\vec{F}_G\right|} \quad (13)$$

If $m_I \neq m_G$, this is a bit of a mess, but if we take $m_I = m_G$ to get an idea of the size of the angle, we have for the latitude of Seattle (47° so $\theta = 90° - 47° = 43°$) (using Maple to do the algebra):

$$\cos\alpha = 0.9999985348 \quad (14)$$

which corresponds to an angle of

$$\alpha = 0.001712 \text{ rad} \quad (15)$$

or just under a tenth of a degree. Although small, the angle should be easily measurable in an accurate experiment.

The point of this setup is that the gravitational force $\vec{F}_G$ has a small component that is perpendicular to $\vec{T}$ so that if m_I/m_G is different for the two weights, there will be an imbalance in the transverse forces acting on the two weights, which will cause the dumbbell to twist around its axis. The component of $\vec{F}_G$ transverse to the fibre is

$$\vec{F}_G \sin\alpha \approx 9.8 \times \alpha \times m_G = 0.0168 m_G \text{ Newtons} \quad (16)$$

If the two types of mass are different, then we have for the transverse forces on the two weights (which we'll refer to as A and B):

$$\begin{aligned} m_{I,A} a_{t,A} &= m_{G,A} g_t \\ m_{I,B} a_{t,B} &= m_{G,B} g_t \end{aligned} \quad (17)$$

where the subscript t indicates the transverse acceleration. The value of g_t is the same for both masses since it is the component of the gravitational acceleration $\vec{g}$ which is always 9.8 m s^{-2}. If the inertial and gravitational masses are not in the same ratio for the two weights, then

$$\frac{a_{t,A}-a_{t,B}}{\frac{1}{2}\left(a_{t,A}+a_{t,B}\right)}=2\left[\frac{m_{G,A}}{m_{I,A}}-\frac{m_{G,B}}{m_{I,B}}\right]\left[\frac{m_{G,A}}{m_{I,A}}+\frac{m_{G,B}}{m_{I,B}}\right]^{-1} \tag{18}$$

If $\frac{m_{G,A}}{m_{I,A}}=\frac{m_{G,B}}{m_{I,B}}$, then this quantity is zero, and the two masses experience the same transverse force so there is no twisting of the fibre.

The actual experiment is considerably more involved than this model, as it had to account for the fact that the Earth isn't a perfect sphere, and that the presence of other masses in the vicinity of the apparatus would affect the results, and so on. However, the net result is that $\frac{m_{G,A}}{m_{I,A}}=\frac{m_{G,B}}{m_{I,B}}$ to an accuracy of 1.5×10^{-13}, so it seems safe to conclude that inertial and gravitational mass are the same (or at least their ratio is constant for the substances tested).