ESKIMO MITES AND THEIR METRIC

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Zee's Einstein Gravity book considers a fictional race of mites that live near the north pole of a unit sphere (radius = 1). They use the north pole as their origin and consider only short distances from this origin. The usual cartesian coordinates in this case become

$$x = \sin\theta\cos\phi \approx \theta\cos\phi \tag{1}$$

$$y = \sin\theta\sin\phi \approx \theta\sin\phi \tag{2}$$

[Note that we're using the usual spherical coordinates where $\theta = 0$ at the north pole, rather than latitude and longitude.]

The mites have worked out that their metric, to second order, is given by

$$ds^2 = \left(1 - \frac{y^2}{3}\right)dx^2 + \left(1 - \frac{x^3}{3}\right)dy^2 + \frac{2}{3}xy\,dxdy + \ldots \tag{3}$$

We need to show that this is equivalent to the ordinary metric in spherical coordinates which is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{4}$$

from which we get

$$\begin{aligned} dx &= \cos\phi\,d\theta - \theta\sin\phi\,d\phi \\ dy &= \sin\phi\,d\theta + \theta\cos\phi\,d\phi \end{aligned} \tag{5}$$

We can solve for $d\theta$ and $d\phi$ to get

$$d\theta = \cos\phi\,dx + \sin\phi\,dy \tag{6}$$

$$d\phi = \frac{1}{\theta}(\cos\phi\,dy - \sin\phi\,dx) \tag{7}$$

Plugging this into 4 and expanding, we have

$$ds^2 = (\cos\phi\,dx + \sin\phi\,dy)^2 + \frac{\sin^2\theta}{\theta^2}(\cos\phi\,dy - \sin\phi\,dx)^2 \tag{8}$$

We can now approximate

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} \tag{9}$$

and keep only terms up to θ^2 in the expansion of $\frac{\sin^2\theta}{\theta^2}$, along with $\cos^2\phi + \sin^2\phi = 1$ to get

$$\begin{aligned} ds^2 &= dx^2 + dy^2 - 2\frac{\theta^2}{3!}\left(\cos\phi\, dy - \sin\phi\, dx\right)^2 && (10)\\ &= \left(1 - \frac{1}{3}\left(\theta\sin\phi\right)^2\right)dx^2 + \left(1 - \frac{1}{3}\left(\theta\cos\phi\right)^2\right)dy^2 + \frac{2}{3}\theta^2\sin\phi\cos\phi && (11)\\ &= \left(1 - \frac{y^2}{3}\right)dx^2 + \left(1 - \frac{x^3}{3}\right)dy^2 + \frac{2}{3}xy\, dxdy && (12) \end{aligned}$$

where we used 1 to get the last line.

We could also do this by starting with 1. We plug 1 and 5 into 3 and expand. I used Maple to do this, as it's somewhat messy, with the result

$$ds^2 = d\theta^2 + \frac{1}{3}\left(3\theta^2 - \theta^4\right)d\phi^2 \tag{13}$$

To first order in θ, $\sin\theta \approx \theta$ and we can ignore the θ^4 term, so we get 4, which verifies the result.

If you really want to grind through the calculations by hand, here is what Maple gives for intermediate stages in the calculation.

$$\left(1 - \frac{y^2}{3}\right)dx^2 = -\frac{1}{3}\left(\theta^2\cos^2\phi - \theta^2 + 3\right)\times \left(\theta^2\cos^2\phi d\phi^2 + 2\theta\cos\phi\sin(\phi)\, d\theta d\phi - \cos^2\phi d\theta^2 - \theta^2 d\phi^2\right) \tag{14}$$

$$\left(1 - \frac{x^3}{3}\right)dy^2 = -\frac{1}{3}\left(\theta^2\cos^2\phi - 3\right)\times \left(\theta^2\cos^2\phi d\phi^2 + 2\theta\cos\phi\sin(\phi)\, d\theta d\phi - \cos^2\phi d\theta^2 + d\theta^2\right) \tag{15}$$

$$\frac{2}{3}xy\, dxdy = -\frac{2}{3}\theta^2\cos(\phi)\sin(\phi)\times \left(-2\theta\cos^2\phi\, d\phi d\theta + \sin\phi\cos\phi\left(\theta^2 d\phi^2 - d\theta^2\right) + \theta d\phi d\theta\right) \tag{16}$$

When we add up these three terms and simplify, we get 13.