FALLING INTO A BLACK HOLE

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Here are a couple of examples of calculating proper time for an object crossing the event horizon. If the object starts at rest at $r=R$ and then falls to $r=r_0$, the proper time interval is

$$\Delta\tau=\sqrt{\frac{R}{8GM}}\left[2\sqrt{r(R-r)}+R\arctan\left(\frac{R-2r}{2\sqrt{r(R-r)}}\right)\right]\Bigg|_{r_0}^{R} \tag{1}$$

An alternative form of this is found by using the identity:

$$\arctan x=\arcsin\frac{x}{\sqrt{1+x^2}} \tag{2}$$

A bit of algebra produces:

$$\left(1+\left[\frac{R-2r}{2\sqrt{r(R-r)}}\right]^2\right)^{-1/2}=\left[\frac{4\left(rR-r^2\right)+R^2-4rR+4r^2}{4\left(rR-r^2\right)}\right]^{-1/2} \tag{3}$$

$$=\left(\frac{R^2}{4r(R-r)}\right)^{-1/2} \tag{4}$$

$$=\frac{2\sqrt{r(R-r)}}{R} \tag{5}$$

With $x=\frac{R-2r}{2\sqrt{r(R-r)}}$ in 2 we therefore have

$$\left(\frac{R-2r}{2\sqrt{r(R-r)}}\right)\left(1+\left[\frac{R-2r}{2\sqrt{r(R-r)}}\right]^2\right)^{-1/2}=\left(\frac{R-2r}{2\sqrt{r(R-r)}}\right)\frac{2\sqrt{r(R-r)}}{R} \tag{6}$$

$$=\frac{R-2r}{R} \tag{7}$$

Therefore

$$\Delta\tau = \sqrt{\frac{R}{8GM}}\left[2\sqrt{r(R-r)}+R\arcsin\left(\frac{R-2r}{R}\right)\right]\Bigg|_{r_0}^{R} \tag{8}$$

$$= \sqrt{\frac{R}{8GM}}\left(R\arcsin(-1)-2\sqrt{r_0(R-r_0)}-R\arcsin\left(\frac{R-2r_0}{R}\right)\right) \tag{9}$$

$$= \sqrt{\frac{R}{8GM}}\left(\frac{3\pi R}{2}-\sqrt{r_0(R-r_0)}-R\arcsin\left(\frac{R-2r_0}{R}\right)\right) \tag{10}$$

Example 1: If we start at radius of $R = 10GM$ outside a black hole with a million solar masses, then

$$GM = 10^6 (GM)_{\odot} \tag{11}$$

$$= 1.477 \times 10^6 \text{ km} \tag{12}$$

The time to fall to zero radius ($r_0 = 0$) is then

$$\Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}} \tag{13}$$

$$= \pi\sqrt{\frac{1000}{8}}GM \tag{14}$$

$$= 5.188 \times 10^7 \text{ km} = 173.1 \text{ sec} \tag{15}$$

Example 2: We now start at a radius of $r_1 = 16GM$ from the same black hole, and launch an object radially outwards with a speed such that it comes to rest at $r_2 = 32GM$. The object then falls radially inwards until it crosses the event horizon at $r_3 = 2GM$ and then continues until it reaches $r_4 = 0$. We can work out the times for each leg of the journey separately. We can get these times by inserting the appropriate limits in 10.

For the first leg, the time is the same as if the object started at $R = 32GM$ and fell to $r_0 = 16GM$. This comes out to

$$\Delta\tau_{1\to 2} = 32(2+\pi)GM = 2.43 \times 10^8 \text{ km} = 810.6 \text{ sec} \tag{16}$$

Next, the object falls from rest at $R = 32GM$ to the event horizon at $r_0 = 2GM$:

$$\Delta\tau_{2\to 3} = 8\left(\sqrt{15}+8\arcsin\left(\frac{7}{8}\right)+4\pi\right)GM \tag{17}$$

$$= 199.7GM = 2.9496 \times 10^8 \text{ km} = 983.89 \text{ sec} \tag{18}$$

Finally, the object falls from $R=2GM$ to $r_0=0$. However, it is not starting from rest so we can't use 10 directly. We can find the total time taken to fall from $R=32GM$ to $r_0=0$ and then take the difference. The total time is

$$\Delta\tau_{32GM\to 0} = \frac{\pi(32GM)^{3/2}}{\sqrt{8GM}} \tag{19}$$
$$= 64\pi GM \tag{20}$$
$$= 2.96968\times 10^8 \text{ km} \tag{21}$$
$$= 990.58 \text{ sec} \tag{22}$$

The time to fall from the event horizon at $2GM$ to 0 is therefore

$$\Delta\tau_{3\to 4} = 990.58 - 983.89 = 6.69 \text{ sec} \tag{23}$$

The exact answer depends on how accurate the value of the speed of light is. I've used $c = 299792$ km s^{-1}. The answer given in Moore's student manual is 7.5 sec which presumably arises from round off errors in calculating the various $\Delta\tau$s.