

## FOUR-MOMENTUM CONSERVATION IN A SPACESHIP

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As a rather fanciful example of using the conservation of four-momentum, suppose we have a spaceship of total mass (including fuel)  $M$ , initially at rest on Earth. The fuel consists of matter/anti-matter which when mixed, produces photons that are ejected out of the back of the ship. The ship burns enough fuel to accelerate to  $v = 0.95$ , then travels to some star system such as Alpha Centauri, then decelerates to zero for a landing. After some time at its destination it reverses the trip by again accelerating to  $v = 0.95$ , returning to Earth, and decelerating to rest. What is its final mass as a fraction of its initial mass?

We can assume that all motion takes place along the  $x$  axis, and treat the problem in the Earth's frame. Then the initial momentum is

$$\mathbf{p}_0 = [M, 0] \quad (1)$$

After accelerating to its final speed of  $v = 0.95$ , the combined momentum of the ship + ejected photons is

$$\mathbf{p}_1 = \gamma m_1 [1, 0.95] + E_1 [1, -1] \quad (2)$$

where  $\gamma = 1/\sqrt{1-v^2} = 3.2$ ,  $E_1$  is the energy of the ejected photons and  $m_1$  is the mass of the ship after burning the fuel needed to accelerate.

By the conservation of momentum, we have  $\mathbf{p}_1 = \mathbf{p}_0$  so

$$\gamma m_1 + E_1 = M \quad (3)$$

$$0.95\gamma m_1 - E_1 = 0 \quad (4)$$

Adding these 2 equations, we get

$$m_1 = \frac{M}{1.95\gamma} = 0.16M \quad (5)$$

Now to decelerate the ship, we eject the photons ahead of the ship, and we start with a momentum of  $\gamma m_1 [1, 0.95]$ . After deceleration, the mass is now  $m_2$  and the ship is at rest, so we must have

$$\mathbf{p}_2 = [m_2, 0] + E_2 [1, 1] \quad (6)$$

where  $E_2$  is the total energy of the photons ejected to decelerate the ship.

Again, conservation of momentum requires  $\mathbf{p}_2 = \gamma m_1 [1, 0.95]$ , so

$$\gamma m_1 = m_2 + E_2 \quad (7)$$

$$0.95\gamma m_1 = E_2 \quad (8)$$

Subtracting these equations we get

$$m_2 = 0.05\gamma m_1 \quad (9)$$

$$= \frac{0.05}{1.95} M \quad (10)$$

On the return trip, we go through exactly the same procedure, except we now start with a mass  $m_2$  rather than  $M$ . Thus on the return to Earth, the ship's mass will be

$$m_E = \frac{0.05}{1.95} m_2 \quad (11)$$

$$= \left(\frac{0.05}{1.95}\right)^2 M \quad (12)$$

Thus the ship's initial mass is

$$M = 1521 m_E \quad (13)$$

Virtually all the initial mass is fuel.

Incidentally, if we do this calculation for an arbitrary velocity  $v$ , we get

$$m_1 = \frac{M}{(1+v)\gamma} \quad (14)$$

$$= \sqrt{\frac{1-v}{1+v}} M \quad (15)$$

$$m_2 = \frac{1-v}{1+v} M \quad (16)$$

$$m_E = \left(\frac{1-v}{1+v}\right)^2 M \quad (17)$$

Thus each acceleration or deceleration multiplies the previous mass by a factor of  $\sqrt{\frac{1-v}{1+v}}$ .