FOUR-MOMENTUM CONSERVATION

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A logical extension of the four-velocity **u** is the four-momentum $\mathbf{p}\equiv m\mathbf{u}$, where m is the object's rest mass. From the definition of **u** this means that in an inertial frame in which the object's velocity is **v**:

$$\mathbf{p}=\gamma m\left[1,v_x,v_y,v_z\right] \tag{1}$$

where $\gamma=1/\sqrt{1-v^2}$. Experimentally, it is known that the four-momentum of a collection of particles is conserved, and that the old Newtonian law of conservation of 3-momentum is *not* valid for velocities approaching that of light. As a simple example of this, suppose we have two particles with the same mass travelling towards each other (as viewed in an inertial frame we'll call the rest frame), so that one particle has a speed of $v_1=+v$ and the other has $v_2=-v$. In Newtonian physics, the total momentum of the system is $mv_1-mv_2=0$. Now suppose that the particles collide head-on and that the collision is totally inelastic so that the two particles merge to form a single particle of mass $2m$ with a speed of $v_3=0$. Clearly the 3-momentum after the collision is still zero, so 3-momentum is conserved here.

Now if we observe this collision from an inertial frame moving with a speed v then, before the collision one particle is at rest so that $v_1'=0$. To find the velocity of the other, we need to use the relativistic velocity summation formula to get

$$\begin{aligned}
v_2' &= -\frac{v_2+v}{1+v_2v} && (2)\\
&= -\frac{2v}{1+v^2} && (3)
\end{aligned}$$

After the collision, the single particle is moving at speed $v_3'=-v$. If we use the same mass for the particles in this case, the Newtonian law would be violated since

$$mv_1' + mv_2' = 0 - \frac{2mv}{1+v^2} \tag{4}$$
$$\neq 2mv_3' \tag{5}$$

What happens if we look at four-momentum instead? In the rest frame the four momentum is

$$\mathbf{p} = \gamma m\,[1, v, 0, 0] + \gamma m\,[1, -v, 0, 0] \tag{6}$$
$$= \gamma m\,[2, 0, 0, 0] \tag{7}$$

Since the first component of **p** is the energy (mass) of the particle, we see that in the rest frame, where the compound particle is at rest, its rest mass is

$$M = 2\gamma m \tag{8}$$

if we're assuming four-momentum is conserved.

Now in the moving frame, the four-momentum is

$$\mathbf{p}' = m\,[1, 0, 0, 0] + \gamma' m[1, v_2', 0, 0] \tag{9}$$

since the first particle is at rest and the second is now moving with speed v_2'. We can write γ' in terms of v:

$$\gamma' = \frac{1}{\sqrt{1-\left(v_2'\right)^2}} \tag{10}$$
$$= \left(1 - \frac{4v^2}{(1+v^2)^2}\right)^{-1/2} \tag{11}$$
$$= \frac{1+v^2}{\sqrt{(1+v^2)^2 - 4v^2}} \tag{12}$$
$$= \frac{1+v^2}{\sqrt{1-2v^2+v^4}} \tag{13}$$
$$= \frac{1+v^2}{1-v^2} \tag{14}$$

Therefore, the total momentum in the moving frame is

$$\mathbf{p}' = m[1,0,0,0] + \frac{1+v^2}{1-v^2}m[1,-\frac{2v}{1+v^2},0,0] \tag{15}$$
$$= 2m\gamma^2[1,-v,0,0] \tag{16}$$
$$= \gamma M[1,-v,0,0] \tag{17}$$

But this is also the momentum of a particle of rest mass M moving with a speed v, so in this case, four-momentum is indeed conserved.

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