FOUR-VECTORS - SUMMATION CONVENTION

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Here are a few examples of dealing with the summation convention applied to four-vectors, henceforth referred to as just 'vectors'.

The summation convention states that if the same index appears once up and once down in the same expression, it should be summed. Greek indices range from 0 to 3, Roman indices from 1 to 3.

Example 1. An example illustrating the summation convention. We are given

$$A^{\mu} = [5,0,-1,-6] \tag{1}$$
$$B_{\nu} = [0,-2,4,0] \tag{2}$$
$$C_{0\alpha} = [1,0,2,3] \tag{3}$$
$$C_{1\alpha} = [5,-2,-2,0] \tag{4}$$
$$C_{2\alpha} = [4,5,2,-2] \tag{5}$$
$$C_{3\alpha} = [-1,-1,-3,0] \tag{6}$$

Note the relative positions of the indices (upper or lower) in each case. Thus we have

(1) $A^{\alpha}B_{\alpha} = 5\times 0+0\times(-2)-1\times 4-6\times 0 = -4$.

(2) $A^{\alpha}C_{\alpha\beta}$ for all β. For $\beta=0$, we have $5+0-4+6=15$. For $\beta=1$, $0+0-5+6=1$. For $\beta=2$, $10+0-2+18=26$. For $\beta=3$, $15+0+2+0=17$.

(3) $A^{\gamma}C_{\gamma\sigma}$ for all σ. This is the same as (2) with the indices relabelled. The letters used for labels don't matter as long as the same pattern occurs.

(4) $A^{\nu}C_{\mu\nu}$. This time we sum over the *second* index in $C_{\mu\nu}$, so we have: $\mu=0$, $A^{\nu}C_{0\nu}=-15$; $\mu=1$, $A^{\nu}C_{1\nu}=27$; $\mu=2$, $A^{\nu}C_{2\nu}=30$; $\mu=3$, $A^{\nu}C_{3\nu}=-2$.

(5) $A^{\alpha}B_{\beta}$ for all α, β. Each term is a product of two numbers, without any summation. So we have $A^0B_0=0$, $A^0B_1=-10$ and so on.

(6) A^iB_i. Since Roman indices are used instead of Greek, the sum is over $i=1,2,3$, with the result -4.

(7) $A^j B_k$ for all j,k. Same as (5) except now we consider only $j,k \in \{1,2,3\}$.

Example 2. Free versus dummy indices. A pair of dummy indices is summed over, while free indices must match on both sides of an equation.

(1) $A^\alpha B_\alpha = 5$. α is a dummy index.
(2) $A^{\overline{\mu}} = \Lambda^{\overline{\mu}}{}_\nu A^\nu$. $\overline{\mu}$ is a free index; ν is a dummy index.
(3) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha}$. μ and λ are dummies; γ and α are free.
(4) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}$. Both indices are free; no sums.