

## FOUR-VELOCITY - AN EXAMPLE

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We've already looked at the four-velocity in special relativity, but it's worth a second look from a different angle. We can instead define the four-velocity in terms of two events separated by an infinitesimal spacetime interval  $ds$ . The four-velocity is defined as the derivative of  $ds$  with respect to the proper time  $\tau$ , so that

$$u^i \equiv \frac{ds^i}{d\tau} \quad (1)$$

Since the components  $ds^i$  transform using the Lorentz transformation, then so do the components  $u^i$  of the four-velocity.

Since the spacetime interval is invariant (it has the same value in all inertial frames) the relation

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (2)$$

holds in all inertial frames. In particular, it holds in the observer's rest frame, in which  $dt = d\tau$ , so we have

$$ds^2 = -d\tau^2 \quad (3)$$

In this rest frame, then, we get

$$u^i = \left( \frac{d\tau}{d\tau}, 0, 0, 0 \right) \quad (4)$$

$$= (1, 0, 0, 0) \quad (5)$$

The square of  $\mathbf{u}$  is then

$$\mathbf{u} \cdot \mathbf{u} = \eta_{ij} u^i u^j \quad (6)$$

$$= -1 \quad (7)$$

where  $\eta_{ij}$  is the metric used in special relativity:

$$\eta_{ij} = \begin{cases} -1 & i = j = t \\ 1 & i = j = x, y, z \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Since this is true in the rest frame and the square of a four-vector is an invariant, it is true in all frames.

As an example, suppose we have an object that moves along a worldline given by (in some inertial frame)

$$x(\tau) = \frac{1}{g} [\cosh(g\tau) - 1] \quad (9)$$

( $y = z = 0$  and  $g$  is a constant). The  $x$  component of the four-velocity is then

$$u^x = \frac{dx}{d\tau} \quad (10)$$

$$= \sinh(g\tau) \quad (11)$$

Using  $\mathbf{u} \cdot \mathbf{u} = -1$  we can find the  $t$  component:

$$(u^x)^2 - (u^t)^2 = -1 \quad (12)$$

$$\sinh^2(g\tau) + 1 = (u^t)^2 \quad (13)$$

$$u^t = \cosh(g\tau) \quad (14)$$

using the identity  $\cosh^2 x - \sinh^2 x = 1$ . From this we can get the time in the inertial frame:

$$u^t = \frac{dt}{d\tau} \quad (15)$$

$$t(\tau) = \frac{1}{g} \sinh(g\tau) \quad (16)$$

The velocity of the object as seen in the inertial frame is

$$v_x = \frac{dx}{dt} \quad (17)$$

$$= \frac{u^x}{u^t} \quad (18)$$

$$= \tanh(g\tau) \quad (19)$$

Since  $\tanh$  is bounded by  $\pm 1$ , the velocity never exceeds 1, so never exceeds the speed of light.

We can invert the relation between proper time  $\tau$  and inertial time  $t$  to get

$$g\tau = \sinh^{-1}(gt) \quad (20)$$

Using the relations (derived from  $\cosh^2 x - \sinh^2 x = 1$ )

$$\sinh(\sinh^{-1} x) = x \quad (21)$$

$$\cosh(\sinh^{-1} x) = \sqrt{1 + x^2} \quad (22)$$

$$\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1 + x^2}} \quad (23)$$

we get

$$u^x(t) = gt \quad (24)$$

$$u^t(t) = \sqrt{1 + (gt)^2} \quad (25)$$

$$v(t) = \frac{gt}{\sqrt{1 + (gt)^2}} \quad (26)$$

Again, note that  $\mathbf{u} \cdot \mathbf{u} = -1$  and also that as  $t \rightarrow \infty$ ,  $v \rightarrow 1$  so again the velocity remains less than  $c$ .