

FOUR-VELOCITY - ANOTHER EXAMPLE

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Another example of four-velocity in special relativity. We start with an object whose velocity (3-d velocity, that is) in an inertial frame is

$$v_x(t) = \sqrt{1 - \frac{1}{(gt+1)^2}} \quad (1)$$

where g is a constant with relativistic units of m^{-1} , and $t \geq 0$ is the time measured in the inertial frame (so it's not the proper time of the object). The suffix x indicates the motion is along the x axis, as usual.

The relation between a proper time interval $d\tau$ and the time interval dt measured in a frame moving at speed v_x along the x axis with respect to the object is given by the time dilation formula

$$d\tau = dt \sqrt{1 - v_x^2} \quad (2)$$

This gives us the time component of the four-velocity:

$$u^t = \frac{dt}{d\tau} \quad (3)$$

$$= \frac{1}{\sqrt{1 - v_x^2}} \quad (4)$$

$$= 1 + gt \quad (5)$$

We can integrate this to get τ in terms of t :

$$\frac{dt}{1 + gt} = d\tau \quad (6)$$

$$\frac{1}{g} \ln(1 + gt) = \tau + \tau_0 \quad (7)$$

where τ_0 is the constant of integration. If we require $\tau = 0$ when $t = 0$, then $\tau_0 = 0$ and we get

$$g\tau = \ln(1 + gt) \quad (8)$$

From this we get

$$u^t = e^{g\tau} \quad (9)$$

From the definition of four-velocity, we have

$$u^x \equiv \frac{dx}{d\tau} \quad (10)$$

$$= \frac{dx}{dt \sqrt{1 - v_x^2}} \quad (11)$$

$$= \frac{v_x}{\sqrt{1 - v_x^2}} \quad (12)$$

$$= (1 + gt) \sqrt{1 - \frac{1}{(gt + 1)^2}} \quad (13)$$

$$= \sqrt{(1 + gt)^2 - 1} \quad (14)$$

$$= \sqrt{e^{2g\tau} - 1} \quad (15)$$

We can now find x and t as functions of τ :

$$t(\tau) = \frac{1}{g} (e^{g\tau} - 1) \quad (16)$$

$$x(\tau) = \int_0^\tau \sqrt{e^{2g\tau'} - 1} d\tau' \quad (17)$$

$$= \frac{1}{g} \left[\sqrt{e^{2g\tau} - 1} - \arctan \left(\sqrt{e^{2g\tau} - 1} \right) \right] \quad (18)$$

using Maple to do the integral.