FOUR-VELOCITY - ANOTHER EXAMPLE

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Another example of four-velocity in special relativity. We start with an object whose velocity (3-d velocity, that is) in an inertial frame is

$$v_x(t) = \sqrt{1 - \frac{1}{(gt+1)^2}} \tag{1}$$

where g is a constant with relativistic units of m^{-1}, and $t \geq 0$ is the time measured in the inertial frame (so it's not the proper time of the object). The suffix x indicates the motion is along the x axis, as usual.

The relation between a proper time interval $d\tau$ and the time interval dt measured in a frame moving at speed v_x along the x axis with respect to the object is given by the time dilation formula

$$d\tau = dt\sqrt{1 - v_x^2} \tag{2}$$

This gives us the time component of the four-velocity:

$$u^t = \frac{dt}{d\tau} \tag{3}$$

$$= \frac{1}{\sqrt{1 - v_x^2}} \tag{4}$$

$$= 1 + gt \tag{5}$$

We can integrate this to get τ in terms of t:

$$\frac{dt}{1+gt} = d\tau \tag{6}$$

$$\frac{1}{g}\ln(1+gt) = \tau + \tau_0 \tag{7}$$

where τ_0 is the constant of integration. If we require $\tau = 0$ when $t = 0$, then $\tau_0 = 0$ and we get

$$g\tau = \ln(1+gt) \tag{8}$$

From this we get

$$u^t = e^{g\tau} \tag{9}$$

From the definition of four-velocity, we have

$$\begin{aligned}
u^x &\equiv \frac{dx}{d\tau} &(10)\\
&= \frac{dx}{dt\sqrt{1-v_x^2}} &(11)\\
&= \frac{v_x}{\sqrt{1-v_x^2}} &(12)\\
&= (1+gt)\sqrt{1-\frac{1}{(gt+1)^2}} &(13)\\
&= \sqrt{(1+gt)^2-1} &(14)\\
&= \sqrt{e^{2g\tau}-1} &(15)
\end{aligned}$$

We can now find x and t as functions of τ:

$$\begin{aligned}
t(\tau) &= \frac{1}{g}\left(e^{g\tau}-1\right) &(16)\\
x(\tau) &= \int_0^\tau \sqrt{e^{2g\tau'}-1}\,d\tau' &(17)\\
&= \frac{1}{g}\left[\sqrt{e^{2g\tau}-1}-\arctan\left(\sqrt{e^{2g\tau}-1}\right)\right] &(18)
\end{aligned}$$

using Maple to do the integral.

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