FOUR-VELOCITY

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In special relativity, an event is defined by its location in spacetime, which is represented by a four-vector. The components of the event's four-vector depend on the inertial frame in which they are measured, but in general can be represented by three space and one time coordinate:

$$\vec{x} = \left(x^0, x^1, x^2, x^3\right) \tag{1}$$
$$= (t, x, y, z) \tag{2}$$

In an object's rest frame, its time coordinate is the proper time τ and its spatial location remains constant. Thus the interval between two events in the rest frame can be represented by the four-vector

$$\Delta\vec{x} = (\Delta\tau, 0, 0, 0) \tag{3}$$

In Euclidean space, the velocity of an object is the derivative of its location vector with respect to time. We can extend this to spacetime by defining the four-velocity of an object as the derivative of its position four-vector with respect to proper time, so we have

$$\vec{U} \equiv \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \tag{4}$$

or, in index notation:

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{5}$$

In an object's rest frame, the spatial derivatives are zero (because the object is at rest) and $x^0 = \tau$, so in the rest frame

$$\vec{U} = (1, 0, 0, 0) \tag{6}$$

The squared magnitude of $\vec{U}$ is therefore

$$\vec{U} \cdot \vec{U} = -1 \tag{7}$$

Since scalar products are Lorentz invariant, this result applies in all inertial frames.

In some other inertial frame, moving relative to the rest frame, in general all the derivatives are non-zero. However, we know that, due to time dilation, the time interval measured in the rest frame as $d\tau$ is measured in the other frame as $\gamma\, d\tau = dx^0$. Thus we have

$$\frac{dx^0}{d\tau} = \gamma \tag{8}$$

Using the chain rule, we can write the spatial derivatives in 4 as

$$\frac{dx^i}{d\tau} = \frac{dx^i}{dx^0}\frac{dx^0}{d\tau} \tag{9}$$

The first derivative on the RHS is just the object's velocity v^i in the i direction, since x^0 is the time as measured in the moving frame. We therefore have

$$\frac{dx^i}{d\tau} = v^i \frac{dx^0}{d\tau} = \gamma v^i \tag{10}$$

The four-velocity's components in any inertial frame are thus

$$\vec{U} = \gamma\left(1, v^1, v^2, v^3\right) \tag{11}$$

We can verify 7, since

$$\vec{U}\cdot\vec{U} = \gamma^2\left(-1 + \left(v^1\right)^2 + \left(v^2\right)^2 + \left(v^3\right)^2\right) \tag{12}$$

$$= \gamma^2\left(-1 + \boldsymbol{v}^2\right) \tag{13}$$

$$= -1 \tag{14}$$

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