

## GEODESIC EQUATION - GEODESICS ON A SPHERE

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We've seen that geodesic curves are found by maximizing the proper time between two events, where the proper time interval is given by

$$\Delta\tau = \int_0^1 \sqrt{-g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} d\sigma \quad (1)$$

We can do this by solving the set of Lagrangian differential equations given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) - \frac{\partial L}{\partial q_a} = 0 \quad (2)$$

where the Lagrangian function is given by

$$L = \sqrt{-g_{ij}(x^k(\sigma)) \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} \quad (3)$$

Here, the object's world path is given in parametric form by  $x^i(\sigma)$ . The metric can depend on the position, so it too is ultimately a function of the parameter  $\sigma$ , which plays the role of  $t$  in 2. For this Lagrangian, the coordinates are  $q_a = x^a$  and the generalized velocities are  $\dot{q}_a = \frac{dx^a}{d\sigma} \equiv \dot{x}^a$ . We therefore have

$$\frac{\partial L}{\partial \dot{x}^a} = -\frac{1}{2L} g_{ij} \left( \delta^i_a \frac{dx^j}{d\sigma} + \frac{dx^i}{d\sigma} \delta^j_a \right) \quad (4)$$

$$= -\frac{g_{aj}}{L} \frac{dx^j}{d\sigma} \quad (5)$$

This follows because  $g_{ij}$  does not depend on  $\dot{x}^a$  (it depends only on the coordinates, not the velocities), and also because  $g_{aj} = g_{ja}$ .

The other derivative comes out to

$$\frac{\partial L}{\partial x^a} = -\frac{1}{2L} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \quad (6)$$

We can get rid of  $L$  in these two results by noticing from 1 that

$$\Delta\tau = \int_{\tau_0}^{\tau_1} d\tau \quad (7)$$

$$= \int_0^1 \frac{d\tau}{d\sigma} d\sigma \quad (8)$$

so

$$\frac{d\tau}{d\sigma} = \sqrt{-g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} = L \quad (9)$$

We therefore get from 5 and 6

$$\frac{\partial L}{\partial x^a} = -\frac{g_{aj}}{L} \frac{dx^j}{d\sigma} \quad (10)$$

$$= -g_{aj} \frac{d\sigma}{d\tau} \frac{dx^j}{d\sigma} \quad (11)$$

$$= -g_{aj} \frac{dx^j}{d\tau} \quad (12)$$

$$\frac{\partial L}{\partial x^a} = -\frac{1}{2L} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \quad (13)$$

$$= -\frac{1}{2} \frac{d\sigma}{d\tau} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \quad (14)$$

$$= -\frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \quad (15)$$

Putting it all together using 2, we get

$$-\frac{d}{d\sigma} \left( g_{aj} \frac{dx^j}{d\tau} \right) + \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\sigma} = 0 \quad (16)$$

We can now eliminate the parameter  $\sigma$  by multiplying through by  $-\frac{d\sigma}{d\tau}$ :

$$\boxed{\frac{d}{d\tau} \left( g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0} \quad (17)$$

This is one form of the *geodesic equation*, which is a second order ordinary differential equation. (It's an ordinary differential equation despite the appearance of the partial derivative  $\frac{\partial g_{ij}}{\partial x^a}$  because we will know the metric as a function of the coordinates, so this derivative will be known when we set out to solve the ODE.)

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As an example of how we can use this equation to determine geodesics, we'll look at the usual example of the curved 2-d surface of a sphere of

radius  $R$ . The metric for this space is, using the usual spherical coordinates  $\theta$  and  $\phi$

$$g_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & (R \sin \theta)^2 \end{bmatrix} \quad (18)$$

The required derivatives of  $g_{ij}$  are

$$\partial_\theta g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2R^2 \sin \theta \cos \theta \end{bmatrix} \quad (19)$$

$$\partial_\phi g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (20)$$

Then 17 becomes (using  $s$  as the parameter in place of  $\tau$ ) for the index  $a = \theta$ :

$$R^2 \frac{d^2 \theta}{ds^2} - R^2 \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 = 0 \quad (21)$$

$$\frac{d^2 \theta}{ds^2} - \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 = 0 \quad (22)$$

And for  $a = \phi$ :

$$\frac{d}{ds} \left( (R \sin \theta)^2 \frac{d\phi}{ds} \right) = 0 \quad (23)$$

$$\frac{d}{ds} \left( \sin^2 \theta \frac{d\phi}{ds} \right) = 0 \quad (24)$$

The last equation can be integrated once to get

$$\sin^2 \theta \frac{d\phi}{ds} = \frac{k}{R} \quad (25)$$

where  $k$  is a constant with dimensions of length, so that  $k/R$  is dimensionless. Substituting this into the other ODE 22 gives

$$\frac{d^2 \theta}{ds^2} - \left( \frac{k}{R \sin^2 \theta} \right)^2 \sin \theta \cos \theta = 0 \quad (26)$$

$$\frac{d^2 \theta}{ds^2} = \frac{k^2 \cos \theta}{R^2 \sin^3 \theta} \quad (27)$$

We've decoupled the equations, although this latest ODE isn't exactly easy to solve. We can make a bit of progress by observing that in 2-d space, the infinitesimal interval is given by  $ds^2 = g_{ij} dx^i dx^j$ , so

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \left( \frac{ds}{ds} \right)^2 = +1 \quad (28)$$

We substitute for  $g_{ij}$  using 18 and use  $x^1 = \theta$  and  $x^2 = \phi$  to get

$$R^2 \left( \frac{d\theta}{ds} \right)^2 + R^2 \sin^2 \theta \left( \frac{d\phi}{ds} \right)^2 = 1 \quad (29)$$

$$R^2 \left( \frac{d\theta}{ds} \right)^2 + R^2 \sin^2 \theta \left( \frac{k}{R \sin^2 \theta} \right)^2 = 1 \quad (30)$$

$$\frac{d\theta}{ds} = \pm \frac{1}{R} \sqrt{1 - \frac{k^2}{\sin^2 \theta}} \quad (31)$$

$$= \pm \frac{1}{R \sin \theta} \sqrt{\sin^2 \theta - k^2} \quad (32)$$

where we used 25 to get the second line.

Although we could integrate this directly (using software), the answer isn't terribly illuminating. We can take a different approach by rearranging the equation to get

$$ds = \pm \frac{R \sin \theta d\theta}{\sqrt{\sin^2 \theta - k^2}} \quad (33)$$

$$= \pm \frac{R \sin \theta d\theta}{\sqrt{1 - \cos^2 \theta - k^2}} \quad (34)$$

$$= \pm \frac{R \sin \theta d\theta}{\sqrt{a^2 - \cos^2 \theta}} \quad (35)$$

where  $a^2 = 1 - k^2$ . Now we can use the substitution  $u = \cos \theta$  with  $du = -\sin \theta d\theta$  and we get

$$\int ds = \pm R \int \frac{\sin \theta d\theta}{\sqrt{a^2 - \cos^2 \theta}} \quad (36)$$

$$= \mp R \int \frac{du}{\sqrt{a^2 - u^2}} \quad (37)$$

$$s = \mp R \arctan \left( \frac{u}{\sqrt{a^2 - u^2}} \right) + u_0 \quad (38)$$

If we want the path length to be zero at  $\theta = \pi/2$ , this corresponds to  $u = 0$ , so we take  $u_0 = 0$ . This gives

$$\tan \frac{s}{R} = \mp \frac{u}{\sqrt{a^2 - u^2}} \quad (39)$$

Defining  $\psi \equiv s/R$  and squaring both sides, we get, using the trig identity

$$1 + \tan^2 \psi = \sec^2 \psi = \frac{1}{\cos^2 \psi} \quad (40)$$

$$\tan^2 \psi = \frac{u^2}{a^2 - u^2} \quad (41)$$

$$a^2 \tan^2 \psi = u^2 (1 + \tan^2 \psi) \quad (42)$$

$$u = \cos \theta = \pm a \sin \psi \quad (43)$$

Returning to 25, we can now eliminate  $\theta$ , since  $\sin^2 \theta = 1 - \cos^2 \theta = 1 - a^2 \sin^2 \psi$

$$(1 - a^2 \sin^2 \psi) \frac{d\phi}{ds} = \frac{k}{R} \quad (44)$$

$$\frac{d\phi}{ds} = \frac{k}{R(1 - a^2 \sin^2 \frac{s}{R})} \quad (45)$$

Integrating this using Maple gives the rather cryptic result

$$\phi = k \operatorname{arctanh} \left( \sqrt{-1 + a^2} \tan \left( \frac{s}{R} \right) \right) \frac{1}{\sqrt{-1 + a^2}} + \phi_0 \quad (46)$$

We can convert this into something more meaningful if we remember that  $k = \sqrt{1 - a^2}$ , so  $\sqrt{-1 + a^2} = ik$ . Also, since

$$\tanh ix = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \quad (47)$$

$$= i \frac{\sin x}{\cos x} \quad (48)$$

$$= i \tan x \quad (49)$$

the inverse functions are related by

$$\operatorname{arctanh} ix = i \operatorname{arctan} x \quad (50)$$

Putting this together, we get

$$\phi - \phi_0 = \frac{1}{i} i \operatorname{arctan} \left( k \tan \left( \frac{s}{R} \right) \right) \quad (51)$$

$$\tan(\phi - \phi_0) = k \tan \left( \frac{s}{R} \right) \quad (52)$$

We now have equations giving  $\theta$  and  $\phi$  in terms of  $s$ :

$$\begin{aligned}\cos \theta &= \pm a \sin \frac{s}{R} \\ \tan(\phi - \phi_0) &= k \tan\left(\frac{s}{R}\right)\end{aligned}\tag{53}$$

We know that the geodesics on a sphere should be arcs from great circles, that is, arcs from circles formed by the intersection of a plane containing the centre of the sphere with the sphere itself. Note that we're looking for great circles that connect any two points on the sphere, so these circles need not go through the poles. We can define these circles by considering a plane with equation  $z = my$  where  $m$  is a constant, and its intersection with the sphere  $x^2 + y^2 + z^2 = R^2$ . All these planes will contain the  $x$  axis, but we are free to define the  $x$  axis pointing in any direction from the centre of the sphere so we haven't really restricted the solution in any way.

The intersection of the plane and sphere is given by converting to spherical coordinates:

$$x^2 + y^2 + (my)^2 = R^2\tag{54}$$

$$R^2 \sin^2 \theta \cos^2 \phi + (1 + m^2) R^2 \sin^2 \theta \sin^2 \phi = R^2\tag{55}$$

$$\sin^2 \theta (\cos^2 \phi + \sin^2 \phi + m^2 \sin^2 \phi) = 1\tag{56}$$

$$\pm m \sin \phi = \sqrt{\frac{1}{\sin^2 \theta} - 1}\tag{57}$$

$$= \cot \theta\tag{58}$$

Thus the equation  $\pm m \sin \phi = \cot \theta$  is the equation of a great circle that includes the intersection of the  $x$  axis with the sphere. Does this agree with the solution from the geodesic equation above?

Conventional spherical coordinates require  $\phi = 0$  along the  $x$  axis and since we're passing all our planes through that axis, we need to choose the constant  $\phi_0$  above to match this. We've taken  $s = 0$  on the equator, which also intersects the  $x$  axis, so we need to take  $\phi_0 = 0$  as well. Therefore the geodesics 53 are

$$\cos \theta = \pm a \sin \frac{s}{R}\tag{59}$$

$$\tan \phi = k \tan\left(\frac{s}{R}\right)\tag{60}$$

We need to use a bit of trigonometric wizardry to convert these equations. First,  $\sin^2 x = 1 - \cos^2 x = 1 - 1/(1 + \tan^2 x) = \tan^2 x / (1 + \tan^2 x)$ , so

$$\cos^2 \theta = a^2 \frac{\tan^2 \frac{s}{R}}{1 + \tan^2 \frac{s}{R}} \quad (61)$$

$$\frac{1}{1 + \tan^2 \theta} = a^2 \frac{\tan^2 \phi}{k^2 + \tan^2 \phi} \quad (62)$$

$$\frac{\cot^2 \theta}{1 + \cot^2 \theta} = a^2 \frac{\sin^2 \phi}{k^2 (1 - \sin^2 \phi) + \sin^2 \phi} \quad (63)$$

$$= \frac{(1 - k^2) \sin^2 \phi}{k^2 + (1 - k^2) \sin^2 \phi} \quad (64)$$

$$= \frac{\frac{1-k^2}{k^2} \sin^2 \phi}{1 + \frac{1-k^2}{k^2} \sin^2 \phi} \quad (65)$$

The LHS and RHS are equal if

$$\cot^2 \theta = \frac{1 - k^2}{k^2} \sin^2 \phi \quad (66)$$

$$\cot \theta = \pm \sqrt{\frac{1 - k^2}{k^2}} \sin \phi \quad (67)$$

That is, these agree with 58 if we identify

$$m = \sqrt{\frac{1 - k^2}{k^2}} \quad (68)$$

Remember from 25 that  $k$  is a constant of integration, so we can pick it to be anything we like. If we take  $k = 1$ , then  $\cot \theta = 0$  and  $\theta = \frac{\pi}{2}$  so we have the equator of the sphere. Some plots of great circle geodesics for various values of  $m$  are shown in Fig. 1.

These plots were generated in Maple with the commands (for the  $m = 0.25$  plot):

```

cir := arccot(m*sin(theta));
plot3d([[1, theta, phi], [1, theta, subs(m = .25, cir)]], theta
= 0 .. 2*Pi, coords = spherical, view = [-1 .. 1, -1 .. 1,
-1 .. 1], grid = [50, 50], transparency = .9, color = "Salmon",
title = "m = 0.25");
    
```

Using a negative value of  $m$  flips the great circle into the other hemisphere, as we can see in Fig. 2.

## PINGBACKS

Pingback: Geodesic equation and four-velocity

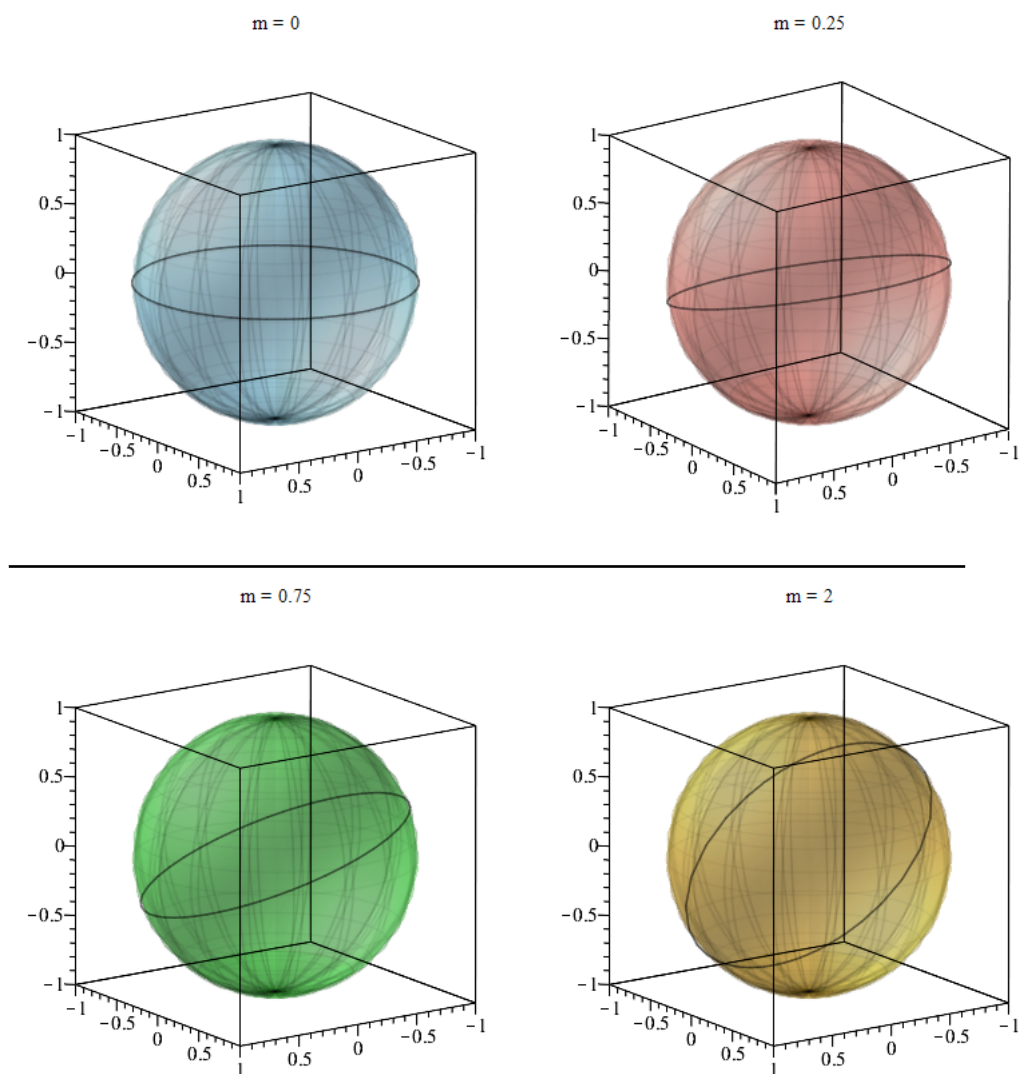


FIGURE 1. Great circle geodesics.

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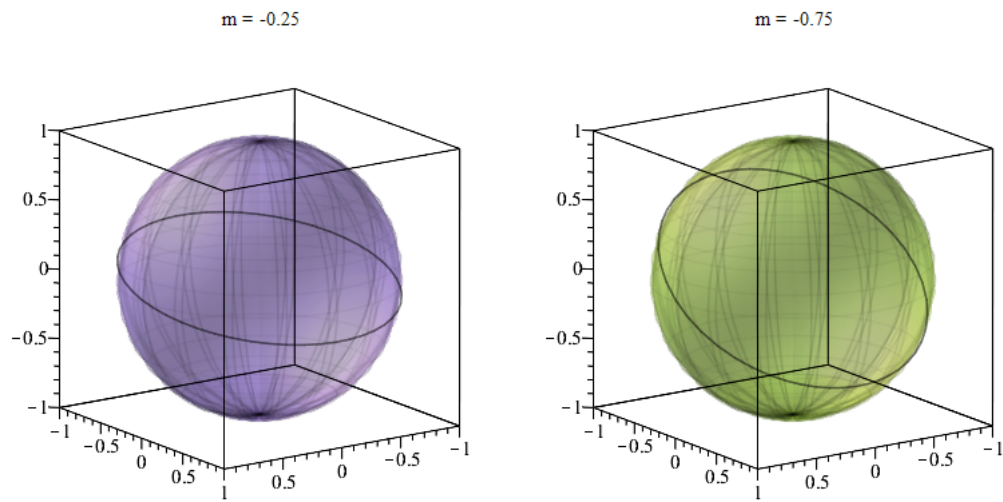


FIGURE 2. Great circle geodesics with negative  $m$ .

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