

GEODESIC EQUATION AND FOUR-VELOCITY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 26 June 2021.

The geodesic equation is

$$\frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \partial_a g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \quad (1)$$

We can write this in terms of the four-velocity $u^i \equiv dx^i/d\tau$ if we expand the first derivative:

$$\frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) = \partial_i g_{aj} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} + g_{aj} \frac{d}{d\tau} \left(\frac{dx^j}{d\tau} \right) \quad (2)$$

$$= \partial_i g_{aj} u^i u^j + g_{aj} \frac{du^j}{d\tau} \quad (3)$$

Therefore, the geodesic equation is

$$\partial_i g_{aj} u^i u^j + g_{aj} \frac{du^j}{d\tau} - \frac{1}{2} \partial_a g_{ij} u^i u^j = 0 \quad (4)$$

We can express this in a different form by multiplying by u^a and summing:

$$u^a \partial_i g_{aj} u^i u^j + g_{aj} u^a \frac{du^j}{d\tau} - \frac{1}{2} u^a \partial_a g_{ij} u^i u^j = g_{aj} u^a \frac{du^j}{d\tau} + \frac{1}{2} u^a u^i u^j \partial_a g_{ij} \quad (5)$$

$$= g_{aj} u^a \frac{du^j}{d\tau} + \frac{1}{2} u^i u^j \frac{dg_{ij}}{d\tau} \quad (6)$$

$$= \frac{1}{2} \frac{d}{d\tau} (g_{ij} u^i u^j) \quad (7)$$

$$= \frac{1}{2} \frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{u}) \quad (8)$$

The geodesic equation thus confirms that $\frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{u}) = 0$ along a geodesic, which we knew beforehand, since $\mathbf{u} \cdot \mathbf{u} = -1$ is an invariant.

PINGBACKS

Pingback: [Geodesic equation - 2-d space-time](#)

Pingback: [Geodesic equation in 2-d with exponential metric](#)