

## GEODESIC EQUATION IN 2-D WITH EXPONENTIAL METRIC

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Post date: 26 June 2021.

This is another example of the geodesic equation in a 2-d space-time (with one time and one space dimension). The metric in this case is

$$ds^2 = -e^{-x/a} dt^2 + dx^2 \quad (1)$$

The metric tensor is then

$$g_{ij} = \begin{bmatrix} -e^{-x/a} & 0 \\ 0 & 1 \end{bmatrix} \quad (2)$$

The geodesic equation is

$$\frac{d}{d\tau} \left( g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \quad (3)$$

Using the time component of the geodesic equation, we get

$$\frac{d}{d\tau} \left( -e^{-x/a} \frac{dt}{d\tau} \right) = 0 \quad (4)$$

$$\frac{1}{a} \frac{dx}{d\tau} e^{-x/a} \frac{dt}{d\tau} - e^{-x/a} \frac{d^2 t}{d\tau^2} = 0 \quad (5)$$

$$\frac{d^2 t}{d\tau^2} = \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau} \quad (6)$$

This might look impossible to solve since we don't know  $dx/d\tau$ , but if we try (for a constant  $c$ ):

$$\frac{dt}{d\tau} = ce^{x/a} \quad (7)$$

$$\frac{d^2 t}{d\tau^2} = \frac{c}{a} \frac{dx}{d\tau} e^{x/a} \quad (8)$$

$$= \frac{1}{a} \frac{dx}{d\tau} \frac{dt}{d\tau} \quad (9)$$

Using the usual trick of requiring  $\mathbf{u} \cdot \mathbf{u} = -1$  we get

$$g_{ij}u^i u^j = -e^{-x/a} \left( \frac{dt}{d\tau} \right)^2 + \left( \frac{dx}{d\tau} \right)^2 \quad (10)$$

$$= -e^{-x/a} c^2 e^{2x/a} + \left( \frac{dx}{d\tau} \right)^2 = -1 \quad (11)$$

$$\frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1} \quad (12)$$

Using the chain rule, we get

$$\frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} \quad (13)$$

$$= c e^{x/a} \frac{dx}{dt} \quad (14)$$

$$\frac{dx}{dt} = \pm \frac{\sqrt{c^2 e^{x/a} - 1}}{c e^{x/a}} \quad (15)$$

$$\pm \int \frac{c e^{x/a}}{\sqrt{c^2 e^{x/a} - 1}} dx = t + t_0 \quad (16)$$

$$\pm \frac{2a}{c} \sqrt{c^2 e^{x/a} - 1} + \alpha = t + t_0 \quad (17)$$

where  $\alpha$  and  $t_0$  are constants of integration. If we set  $t_0 = 0$  and consider  $x$  to be increasing from  $x(0)$ , then we get, by inverting the above equation:

$$x(t) = a \ln \left( \frac{1}{c^2} + \frac{(t - \alpha)^2}{4a^2} \right) \quad (18)$$

At  $t = 0$ , the position is  $x_0$ :

$$x_0 = a \ln \left( \frac{1}{c^2} + \frac{\alpha^2}{4a^2} \right) \quad (19)$$

From 12 and 19, the initial velocity  $\frac{dx}{d\tau}(0) \equiv u_0$  is then

$$u_0 = \sqrt{c^2 e^{x_0/a} - 1} \quad (20)$$

$$= \pm \frac{c\alpha}{2a} \quad (21)$$

We can replace  $\alpha$  by  $u_0$  in the expression for  $x(t)$  to get

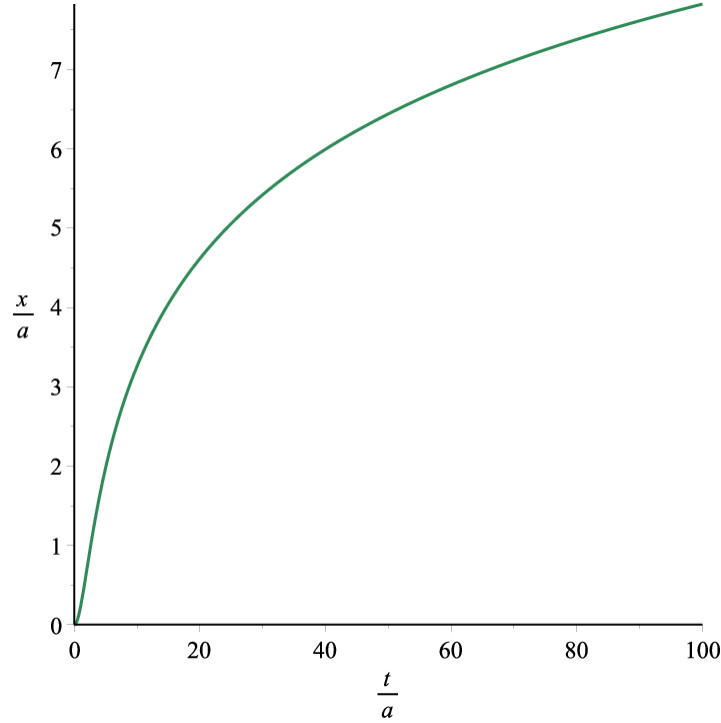


FIGURE 1.

$$x(t) = a \ln \left( \frac{1}{c^2} + \frac{\left(t \pm \frac{2au_0}{c}\right)^2}{4a^2} \right) \quad (22)$$

$$= a \ln \left( \frac{1}{c^2} + \left(\frac{t}{2a} \pm \frac{u_0}{c}\right)^2 \right) \quad (23)$$

If we require  $x_0 = u_0 = 0$  then from above we must have  $a \ln(1/c^2) = 0$ , which gives  $c = \pm 1$  (since  $a$  appears in denominators, it can't be zero). This in turn requires  $\alpha = 0$ , so the geodesic curve is given by

$$x(t) = a \ln \left( 1 + \frac{t^2}{4a^2} \right) \quad (24)$$

If we plot  $\frac{x}{a}$  versus  $\frac{t}{a}$  we get Fig. 1.

We can also work out  $x(\tau)$  and  $t(\tau)$  using the above equations. First, look at  $x$ :

$$\frac{dx}{d\tau} = \pm \sqrt{c^2 e^{x/a} - 1} \quad (25)$$

$$\pm \int \frac{dx}{\sqrt{c^2 e^{x/a} - 1}} = \tau \quad (26)$$

$$\pm 2a \arctan \left( \sqrt{c^2 e^{x/a} - 1} \right) + \beta = \tau \quad (27)$$

where  $\beta$  is a constant of integration. If we require  $\tau_0 = 0$  at  $t = 0$  with the initial conditions above ( $x_0 = u_0 = 0$ ) then using the expression for  $x_0$  above, we find that  $\beta = 0$ . Therefore (taking  $c = 1$ )

$$x(\tau) = a \ln \left( 1 + \tan^2 \frac{\tau}{2a} \right) \quad (28)$$

From this we have from 7 (again, taking  $c = 1$ ):

$$\frac{dt}{d\tau} = e^{x/a} \quad (29)$$

$$= 1 + \tan^2 \frac{\tau}{2a} \quad (30)$$

$$= \sec^2 \frac{\tau}{2a} \quad (31)$$

Integrating we get

$$t(\tau) = \int \sec^2 \frac{\tau}{2a} d\tau \quad (32)$$

$$= 2a \tan \frac{\tau}{2a} \quad (33)$$

where we've chosen the constant of integration so that  $t(0) = 0$ . As  $\tau \rightarrow \pi a$ ,  $t \rightarrow \infty$  so

$$\tau_{max} = \pi a \quad (34)$$