# GLOBALLY PARALLEL VECTOR FIELD

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A vector field $\vec{V}$ is *globally parallel* on a manifold if its covariant derivative is zero on the manifold. The covariant derivative is given in terms of the Christoffel symbols as

$$\left(\nabla\vec{V}\right)^{\alpha}_{\ \ \beta}=V^{\alpha}_{\ ;\beta}=V^{\alpha}_{\ ,\beta}+\Gamma^{\alpha}_{\ \mu\beta}V^{\mu} \tag{1}$$

where semicolon in front of a lower index indicates the covariant derivative and a comma the ordinary derivative.

The condition for $\vec{V}$ to be globally parallel is then

$$V^{\alpha}_{\ ,\beta}+\Gamma^{\alpha}_{\ \mu\beta}V^{\mu}=0 \tag{2}$$

We can convert this into a condition that doesn't involve derivatives of $\vec{V}$ as follows. We take the (ordinary) derivative of 2 with respect to $x^{\nu}$ to get

$$V^{\alpha}_{\ ,\beta\nu}+\Gamma^{\alpha}_{\ \mu\beta,\nu}V^{\mu}+\Gamma^{\alpha}_{\ \mu\beta}V^{\mu}_{\ ,\nu}=0 \tag{3}$$

We can interchange $\beta\leftrightarrow\nu$ to get another equation:

$$V^{\alpha}_{\ ,\nu\beta}+\Gamma^{\alpha}_{\ \mu\nu,\beta}V^{\mu}+\Gamma^{\alpha}_{\ \mu\nu}V^{\mu}_{\ ,\beta}=0 \tag{4}$$

Because the order of derivatives doesn't matter, we have

$$V^{\alpha}_{\ ,\beta\nu}=V^{\alpha}_{\ ,\nu\beta} \tag{5}$$

so taking the difference between 3 and 4 we have

$$\left(\Gamma^{\alpha}_{\ \mu\beta,\nu}-\Gamma^{\alpha}_{\ \mu\nu,\beta}\right)V^{\mu}+\Gamma^{\alpha}_{\ \mu\beta}V^{\mu}_{\ ,\nu}-\Gamma^{\alpha}_{\ \mu\nu}V^{\mu}_{\ ,\beta}=0 \tag{6}$$

We can now eliminate the derivatives of $\vec{V}$ by using 2 in the forms

$$\begin{aligned}V^{\mu}_{\ ,\nu}&=-\Gamma^{\mu}_{\ \sigma\nu}V^{\sigma}\\ V^{\mu}_{\ ,\beta}&=-\Gamma^{\mu}_{\ \sigma\beta}V^{\sigma}\end{aligned} \tag{7}$$

so we get

$$\left(\Gamma^{\alpha}_{\ \mu\beta,\nu}-\Gamma^{\alpha}_{\ \mu\nu,\beta}\right)V^{\mu}=\left(\Gamma^{\alpha}_{\ \mu\beta}\Gamma^{\mu}_{\ \sigma\nu}-\Gamma^{\alpha}_{\ \mu\nu}\Gamma^{\mu}_{\ \sigma\beta}\right)V^{\sigma} \tag{8}$$

We can combine these two terms by swapping $\sigma \leftrightarrow \mu$ on the RHS:

$$\left(\Gamma^{\alpha}{}_{\mu\beta,\nu} - \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\beta}\right) V^{\mu} = 0 \tag{9}$$

This is a necessary condition for a globally parallel vector field. Schutz states that is also a sufficient condition, although we haven't proved it here.