GRADIENT AS A ONE-FORM

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A one-form $\tilde{p}$ is a tensor that acts on a single vector A according to

$$\tilde{p}\left(\vec{A}\right)=A^{\alpha}p_{\alpha} \tag{1}$$

where A^{α} is a component of the vector A and p_{α} is a component of the one-form $\tilde{p}$.

The gradient of a scalar field T (for example, the temperature at various points within a room) satisfies the definition of a one-form. Suppose that T is defined at every spacetime event along some path. The path is one followed by an observer, so we can consider the situation in the frame of the observer. Within this frame, we can label every event along the path with the observer's proper time τ. In some other frame, the coordinates of a particular event are given by the coordinates t, x, y and z. Each of these coordinates can be written as a function of τ, so we have

$$\begin{aligned} t &= t(\tau) \\ x &= x(\tau) \\ y &= y(\tau) \\ z &= z(\tau) \end{aligned} \tag{2}$$

Suppose we now wish to find the rate of change of T along the path of the observer, where by 'rate of change' we mean $dT/d\tau$, that is, the rate of change as viewed by the observer himself. We can calculate this in any frame, by using 2. We have

$$\frac{dT}{d\tau}=\frac{\partial T}{\partial t}\frac{dt}{d\tau}+\frac{\partial T}{\partial x}\frac{dx}{d\tau}+\frac{\partial T}{\partial y}\frac{dy}{d\tau}+\frac{\partial T}{\partial z}\frac{dz}{d\tau} \tag{3}$$

The set of 4 derivatives with respect to τ form the definition of the 4-velocity u:

$$u=\left[\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right] \tag{4}$$

If we write the partial derivatives in 3 as components, we have

$$\tilde{d}T=\left[\frac{\partial T}{\partial t},\frac{\partial T}{\partial x},\frac{\partial T}{\partial y},\frac{\partial T}{\partial z}\right] \tag{5}$$

By comparing this with 1, we see that 3 can be written as

$$\frac{dT}{d\tau}=\tilde{d}T\left(\boldsymbol{u}\right)=u^{\alpha}\left(\tilde{d}T\right)_{\alpha} \tag{6}$$

The quantity $\tilde{d}T$ therefore satisfies the definition of a one-form. From its definition in 5, we see that it is also the usual definition of the gradient (in spacetime). Thus the gradient is a one-form.

In this case, the one-form $\tilde{d}T$ acts on the four-velocity vector $\boldsymbol{u}$ to produce the number $\frac{dT}{d\tau}$, the rate of change of the scalar field (temperature, here) with respect to the proper time of an observer moving along the path. The components of both $\boldsymbol{u}$ and $\tilde{d}T$ depend on the reference frame of the observer making the measurement, but the quantity $\frac{dT}{d\tau}$ is always the same in every reference frame. This is because there is only one frame in which time is measured as the proper time τ, and 6 just provides us with a way to calculate this invariant quantity from *any* reference frame.

The components of $\tilde{d}T$ obey the standard rule for calculating such components. The α component of a one-form $\tilde{d}T$ with respect to a basis vector $\boldsymbol{e}_{\alpha}$ is given by (from 5 and 6):

$$\tilde{d}T_{\alpha}=\tilde{d}T\left(\boldsymbol{e}_{\alpha}\right) \tag{7}$$

$$=\left(\boldsymbol{e}_{\alpha}\right)^{\beta}\tilde{d}T_{\beta} \tag{8}$$

$$=\delta_{\alpha}^{\beta}\tilde{d}T_{\beta} \tag{9}$$

$$=\delta_{\alpha}^{\beta}\frac{\partial T}{\partial x^{\beta}} \tag{10}$$

$$=\frac{\partial T}{\partial x^{\alpha}} \tag{11}$$

An important point is that no metric tensor is needed to produce the result of a one-form acting on a vector. The metric tensor in flat space $\eta_{\mu\nu}$ is a rank two tensor of type $\binom{0}{2}$ which takes two vectors are arguments and produces a number. The scalar product of two vectors must always be defined relative to a metric tensor, so that, in flat space using rectangular coordinates, we have

$$\eta_{\mu\nu}=\left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{12}$$

The scalar product is then defined as

$$\vec{A}\cdot\vec{B} = \eta\left(\vec{A},\vec{B}\right) \tag{13}$$

$$= \eta_{\mu\nu}A^{\mu}B^{\nu} \tag{14}$$

$$= -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \tag{15}$$

The first line is a coordinate-free relation defining the scalar product, while the last two lines write out this scalar product in terms of components, and thus depend on the coordinate system.

The action 1 of a one-form on a vector does *not* require the intervention of a metric, and just sums up the products of the respective components of the vector and one-form. However, the components of a one-form can be obtained from the corresponding components of a vector by using the metric tensor, as we've seen earlier. Thus for a one-form $\tilde{p}$ and a vector $\vec{A}$, the abstract quantity $\tilde{p}\left(\vec{A}\right)$ is a coordinate-free number, but in practice, we usually need to write it out using components to calculate it, and this will often involve deriving the components of the one-form from its corresponding vector (or vice versa).

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