GRADIENT AS COVECTOR - EXAMPLE IN 2-D

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One example of a covariant vector is the gradient. As an example, suppose we have a 2-d scalar field given by $\Phi = bxy = br^2\cos\theta\sin\theta$. In rectangular coordinates

$$\frac{\partial\Phi}{\partial x} = by \tag{1}$$

$$\frac{\partial\Phi}{\partial y} = bx \tag{2}$$

In polar coordinates

$$\frac{\partial\Phi}{\partial r} = 2br\cos\theta\sin\theta \tag{3}$$

$$\frac{\partial\Phi}{\partial\theta} = br^2\left(\cos^2\theta - \sin^2\theta\right) \tag{4}$$

Note that because we have absorbed the factor of r needed for an incremental displacement in the θ direction into the basis vector $\mathbf{e}_\theta$, there is no extra factor of $1/r$ in the $\frac{\partial\Phi}{\partial\theta}$ term, as there would be if we had used unit basis vectors.

Now suppose we have a vector v^i with components given in rectangular coordinates. Then the scalar product is

$$v^i\partial_i\Phi = byv^x + bxv^y \tag{5}$$

If we convert v to polar coordinates, then

$$v^r = v^x\cos\theta + v^y\sin\theta \tag{6}$$

$$v^\theta = -v^x\frac{\sin\theta}{r} + v^y\frac{\cos\theta}{r} \tag{7}$$

The scalar product now is

$$
\begin{aligned}
v^i \partial_i \Phi &= \left(v^x \cos\theta + v^y \sin\theta\right)\left(2br\cos\theta\sin\theta\right) + \\
&\quad \left(-v^x \frac{\sin\theta}{r} + v^y \frac{\cos\theta}{r}\right) br^2 \left(\cos^2\theta - \sin^2\theta\right) && (8) \\
&= brv^x \sin\theta\left(2\cos^2\theta - \cos^2\theta + \sin^2\theta\right) + \\
&\quad brv^y \cos\theta\left(2\sin^2\theta + \cos^2\theta - \sin^2\theta\right) && (9) \\
&= brv^x \sin\theta + brv^y \cos\theta && (10) \\
&= byv^x + bxv^y && (11)
\end{aligned}
$$

Thus the scalar product is invariant.